The model magnetic Laplacian on wedges

Nicolas Popoff

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ABSTRACT. The object of this paper is model Schrödinger operators with constant magnetic fields on infinite wedges with natural boundary conditions. Such model operators play an important role in the semiclassical behavior of magnetic Laplacians on 3d domains with edges. We show that the ground state energy along the wedge is lower than the energy coming from the regular part of the wedge. A consequence of this is the lower semi-continuity of the local ground state energy near an edge for semi-classical Laplacians. We also show that the ground state energy is Hölder with respect to the magnetic field and the wedge aperture, and even Lipschitz when the ground state energy is strictly less than the energy coming from the faces. We finally provide an upper bound for the ground state energy on wedges of small aperture. A few numerical computations illustrate the theoretical approach.

1. INTRODUCTION

1.1. The magnetic Laplacian on model domains.

• *Motivation from the semiclassical problem.* Let $(-ih\nabla - \mathbf{A})^2$ be the magnetic Schrödinger operator (also called the magnetic Laplacian) on an open simply connected subset Ω of $\mathbb{R}^3$. The magnetic potential $\mathbf{A} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ satisfies $\operatorname{curl} \mathbf{A} = \mathbf{B}$ where $\mathbf{B}$ is a regular magnetic field and $h > 0$ is a semiclassical parameter. For Ω bounded with Lipschitz boundary, the operator $(-ih\nabla - \mathbf{A})^2$ assorted with its natural Neumann boundary condition is an essentially self-adjoint operator with compact resolvent. Due to gauge invariance, the spectrum depends on $\mathbf{A}$ only through the magnetic field $\mathbf{B}$.

Many works have been dedicated to understanding the influence of the geometry (defined by the domain Ω and the magnetic field $\mathbf{B}$) on the asymptotics of the first eigenvalue of the magnetic Laplacian and on the localization of the associated eigenfunctions in the semiclassical limit $h \to 0$. When Ω is a two-dimensional polygon and for a non-vanishing magnetic field, the first eigenvalue behaves at first order like $h\mathscr{E}(\mathbf{B}, \Omega)$ where $\mathscr{E}(\mathbf{B}, \Omega) > 0$ is the minimum of the ground state of model magnetic Laplacians (with constant magnetic field) on the plane, the half-plane and infinite sectors, in connection respectively with the interior, the regular parts of the boundary and the corners of Ω (see [5, 31, 22, 19] when Ω is regular and [8, 9, 10] when Ω has corners).

In dimension 3, the regular case is studied in [32, 24, 44], in particular it is proven that the first eigenvalue still has the asymptotic behavior $h\mathscr{E}(\mathbf{B}, \Omega)$ when $h \to 0$ where the constant

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$\mathscr{E}(\mathbf{B},\Omega)$ now involves model problems on the space and the half-space. When the boundary of $\Omega\subset\mathbb{R}^3$ has singularities, only few particular cases have been published and new model magnetic Laplacians associated with the different kind of singularities of the boundary appear. In [38], the domain is a cuboid and the author studies model operators on the octant and on the infinite wedge of opening $\frac{\pi}{2}$ in connection with the corners and the edges of the cuboid. In [43], the authors treat the case of a lens (a domain with an edge that is a closed loop) and a particular orientation of the magnetic field and are led to introduce a model magnetic Laplacian on a infinite wedge with a specific magnetic field.

In all these different cases, the key of success is the study of "local" model magnetic Laplacians on the tangent cones to the boundary and the minimization of their ground state energy along all possible local geometries of Ω. To treat the Schrödinger operator on general 3d domains with edges and (possibly variable) magnetic field, we are led to study the magnetic Laplacian on infinite wedges with constant magnetic field.

Let us add that the main physical motivation for the analysis of the first eigenvalue of the magnetic Laplacian in the semi-classical limit is its applications toward the phenomenon of surface superconductivity for type II superconductors under strong magnetic field (see [20] where a lot of information on the subject can be found). Indeed the asymptotic behavior of the first eigenpairs in the semi-classical limit provides informations on the existence of non-trivial minimizers for the Ginzburg-Landau functional in the large magnetic field limit.

• *The magnetic Laplacian on wedges.* The study of the semi-classical magnetic Laplacian on domains of $\mathbb{R}^3$ with edges involves new model problems on the tangent cones. The tangent cone to an edge is an infinite wedge. Let us denote by (x_1,x_2,x_3) the cartesian coordinates of $\mathbb{R}^3$. Let $\alpha\in(0,\pi)\cup(\pi,2\pi)$ be the opening angle, we denote by $\mathcal{W}_\alpha$ the model wedge of opening α:

$$\mathcal{W}_\alpha:=\mathcal{S}_\alpha\times\mathbb{R} \tag{1.1}$$

where $\mathcal{S}_\alpha$ is the infinite sector defined by $\{(x_1,x_2)\in\mathbb{R}^2,\ |x_2|\leqslant x_1\tan\frac{\alpha}{2}\}$ when $\alpha\in(0,\pi)$ and $\{(x_1,x_2)\in\mathbb{R}^2,\ |x_2|\geqslant x_1\tan\frac{\alpha}{2}\}$ when $\alpha\in(\pi,2\pi)$. We extend these notations by using $\mathcal{W}_\pi$ (respectively $\mathcal{S}_\pi$) for the model half-space (respectively the model half-plane). For $\alpha\neq\pi$ the x_3-axis defines the edge of $\mathcal{W}_\alpha$.

Let $\mathbf{B}$ be a non-zero constant magnetic field and $\mathbf{A}$ an associated linear potential. We define

$$H(\mathbf{A},\,\mathcal{W}_\alpha):=(-i\nabla-\mathbf{A})^2 \tag{1.2}$$

the model magnetic Laplacian on the model domain $\mathcal{W}_\alpha$ with its natural Neumann boundary condition. More precisely the domain of this operator is

$$\{u\in L^2(\mathcal{W}_\alpha),\ (-i\nabla-\mathbf{A})^2u\in L^2(\mathcal{W}_\alpha),\ (-i\nabla-\mathbf{A})u\cdot\mathbf{n}=0\ \text{ on }\ \partial\mathcal{W}_\alpha\}$$

where $\mathbf{n}$ is the outward normal of the boundary $\partial\mathcal{W}_\alpha$ of the wedge (note that $\mathbf{n}$ is well defined almost everywhere). The operator $H(\mathbf{A},\,\mathcal{W}_\alpha)$ is essentially self-adjoint and we denote by

$$E(\mathbf{B},\mathcal{W}_\alpha)\quad\text{the bottom of the spectrum of }\ H(\mathbf{A},\,\mathcal{W}_\alpha)\,. \tag{1.3}$$

Remark 1.1. Due to the elementary scaling $y=|\mathbf{B}|^{1/2}x$, we have $E(\mathbf{B},\mathcal{W}_\alpha)=|\mathbf{B}|E(\frac{\mathbf{B}}{|\mathbf{B}|},\mathcal{W}_\alpha)$ and therefore it is sufficient to consider unitary magnetic fields.

In this article we investigate the bottom of the spectrum of the operator $H(\mathbf{A},\mathcal{W}_\alpha)$ and the influence of the geometry defined by $(\mathbf{B},\alpha)$ with $\mathbf{B}\in\mathbb{S}^2$ on the ground state $E(\mathbf{B},\mathcal{W}_\alpha)$. This operator has already been introduced in particular cases (see subsection 1.4). Our results cover some of these particular cases in a more general context. The consequences of our results on the semiclassical problem on bounded domains are described in Subsection 1.3.

1.2. **Problematics and results.**

• *Tangent substructures of the wedge.* For $\alpha\neq\pi$, the wedge $\mathcal{W}_\alpha$ is a cone of $\mathbb{R}^3$ with *tangent substructures* corresponding to its structure far from its edge. There are three tangent substructures: The half-space Π_α^+ corresponding to the upper face, the half-space Π_α^- corresponding to the lower face and the space $\mathbb{R}^3$ corresponding to interior points. These subsets are linked with the notion of *singular chains* of a cone, see [35] or [17]. When $\alpha\in(0,\pi)$ (convex case) we have $\Pi_\alpha^+=\{(x_1,x_2,x_3)\in\mathbb{R}^3,\ x_2\leqslant x_1\tan\frac{\alpha}{2}\}$ and $\Pi_\alpha^-=\{(x_1,x_2,x_3)\in\mathbb{R}^3,\ x_2\geqslant -x_1\tan\frac{\alpha}{2}\}$. Similar expressions can be found for $\alpha\in(\pi,2\pi)$ (non convex case).

When the model domain is a half-space ($\alpha=\pi$), there is only one tangent substructure: The whole space $\mathbb{R}^3$. The magnetic Laplacian on half-spaces and on $\mathbb{R}^3$ and their ground state energy are naturally defined as in (1.2)-(1.3). On the full space the ground state is well known:

$$\forall\mathbf{B}\in\mathbb{S}^2,\quad E(\mathbf{B},\mathbb{R}^3)=1\ . \tag{1.4}$$

For $\alpha\neq\pi$ we introduce the spectral quantity

$$\mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha):=\min\left\{E(\mathbf{B},\Pi_\alpha^+),E(\mathbf{B},\Pi_\alpha^-),E(\mathbf{B},\mathbb{R}^3)\right\}\ . \tag{1.5}$$

When $\alpha=\pi$, we let $\mathscr{E}^*(\mathbf{B},\mathcal{W}_\pi):=E(\mathbf{B},\mathbb{R}^3)=1$.

• *The operator on half-spaces.* Before describing the meaning of $\mathscr{E}^*$, we recall known result about the magnetic Laplacian on half-spaces and we exhibit the influence of the geometry on $\mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)$. Let $\Pi\subset\mathbb{R}^3$ be a half-space. The bottom of the spectrum of the magnetic Laplacian on Π depends only on the unoriented angle between the magnetic field $\mathbf{B}$ and the boundary of Π. We denote by $\theta\in[0,\frac{\pi}{2}]$ this angle. Let $\sigma(\theta):=E(\mathbf{B},\Pi)$ be the bottom of the spectrum of the operator $H(\mathbf{A},\Pi)$. This function has already been studied in [32], [23] or more recently [13]. In particular $\theta\mapsto\sigma(\theta)$ is increasing over $[0,\frac{\pi}{2}]$ with $\sigma(0)=\Theta_0$ and $\sigma(\frac{\pi}{2})=1$ (see [32]) where the universal constant $\Theta_0\approx 0.59$ is a spectral quantity associated with a unidimensional operator on a half-axis (see [47, 6, 18] and Subsection 2.2).

Let us denote by θ^+ (respectively θ^-) the unoriented angle between the magnetic field $\mathbf{B}$ and Π_α^+ (respectively Π_α^-). We have $E(\mathbf{B},\Pi_\alpha^+)=\sigma(\theta^+)$, $E(\mathbf{B},\Pi_\alpha^-)=\sigma(\theta^-)$ and $E(\mathbf{B},\mathbb{R}^3)=1$. Since σ is increasing we get

$$\mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)=\sigma(\min\{\theta^+,\theta^-\})\ . \tag{1.6}$$

• *Main goals and results.* When $\alpha\neq\pi$, the quantity $\mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)$ can be interpreted as the lowest energy of the magnetic Laplacian far from the edge (x_3-axis). One of the main results of this paper is the following inequality:

$$\forall\alpha\in(0,2\pi),\quad E(\mathbf{B},\mathcal{W}_\alpha)\leqslant\mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)\ , \tag{1.7}$$

roughly speaking that means that the ground state energy associated with an edge is lower than the one of regular adjacent model problems.

Remark 1.2. When $\alpha = \pi$, we have $\theta^- = \theta^+ = \theta$ and $E(\mathbf{B}, \mathcal{W}_\pi) = \sigma(\theta)$. Since $\sigma(\theta) \leqslant 1$ with equality if and only if $\theta = \frac{\pi}{2}$, we notice that inequality (1.7) is already known for $\alpha = \pi$ with equality if and only if $\mathbf{B}$ is normal to the boundary of the half-space $\mathcal{W}_\pi$.

Relation (1.7) may either be strict or be an equality. When inequality (1.7) is strict the singularity makes the energy lower than in the regular cases close to the edge (see Subsection 1.3 for a range of the applications to the semi-classical problem). It has been shown on examples that both cases are possible, see Subsection 1.4. However even in the particular case where the magnetic field is tangent to the edge so that the operator reduces to a pure 2d operator on a sector, the sharp geometrical condition for which (1.7) is strict is only conjectured, see [8, 10]. At this stage, a simple geometrical necessary and sufficient condition for (1.7) to be strict does not seem reachable to us. In Section 5 we will give a sufficient geometrical condition: if the opening angle of the wedge is small enough (depending on $\mathbf{B}$), then (1.7) is strict. This condition may express with analytical functions (see Remark 5.5) and leads to explicit numerical values of the geometrical parameters which ensures that (1.7) is strict.

As we will see, the operator $H(\mathbf{A}, \mathcal{W}_\alpha)$ is fibered: after Fourier transform along the axis of the wedge, it reduces to the family of two-dimensional operators $(\widehat{H}^\tau(\mathbf{A}, \mathcal{W}_\alpha))_{\tau\in\mathbb{R}}$ defined on the sector $\mathcal{S}_\alpha$ (see (2.1)-(2.2)). The operators $(\widehat{H}^\tau(\mathbf{A}, \mathcal{W}_\alpha))_{\tau\in\mathbb{R}}$ are sometimes called the *fibers* of $H(\mathbf{A}, \mathcal{W}_\alpha)$. Its eigenvalues-whenever they exist-seen as functions of τ are called the *band functions*. Their study is the core of the understanding of the spectrum of the magnetic Laplacian on the wedge. By computing both the limit of the first band function and the bottom of the essential spectrum of the fibers, we link $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha)$ and spectral quantities associated with the fibers. As a consequence we will deduce inequality (1.7), moreover when the inequality is strict, we prove the existence of *generalized* eigenpairs for $H(\mathbf{A}, \mathcal{W}_\alpha)$ with energy $E(\mathbf{B}, \mathcal{W}_\alpha)$, moreover these generalized eigenfunctions are localized near the edge (see Corollary 3.8).

Remark 1.3. This kind of analysis of the band functions has its interest for a wider class of fibered operator. This is the case of a two-dimensional Iwatsuka Hamiltonian which is a magnetic Laplacian on $\mathbb{R}^2$ involving a magnetic field $B(x, y) = B(x)$ constant in the y direction, monotonous in the x direction and satisfying $B(x) \to B^\pm$ when $x \to \pm\infty$ (see [26, 33]). The case of a piecewise constant magnetic field is treated in [25] (see also [46] for a physical approach). An analog analysis can be made by setting $\mathscr{E}^* = \min(B^-, B^+)$, that is the ground state energy far from the variation of the magnetic field. The existence of localized (in the x variable) ground state is then given by the analysis of the band functions and depends on whether $E \leqslant \mathscr{E}^*$ is strict or not.

• *Consequences on regularity and positivity of the ground state energy.* The stability of the spectrum of a Schrödinger operator in $\mathbb{R}^3$ under long range perturbation of the magnetic field (this includes perturbation with constant magnetic field) is not described by the standard Kato's perturbation theory and has been the subject of many articles. Under suitable assumptions on the magnetic field and the electric potential, the continuity with respect to the strength of the perturbation has been proved in [4, 36], then in a more general context in [37] and [3]. On one hand, one expects the isolated eigenvalues to have a Lipschitz behavior, on the other hand it is more difficult to study the boundary of the spectrum when it has a band structure (as it is the case here). It is proved implicitly in [36] that the boundary of the band-spectrum is $\frac{1}{2}$-Hölder, the exponent is then pushed to $\frac{2}{3}$ in [14], and recently Cornean has proved in [15] that for constant magnetic field, bands spectrum have Lipschitz stability. Notice that the study of the spectral bands of several Harper-like operators leads to the same stability questions.

In our case perturbations of the magnetic field have a non trivial interaction with the boundary and the results from the above literature do not apply. Moreover we are also interested with perturbation of the geometry of the wedge (that is variation of the aperture angle). The standard resolvent and kernel estimates used in the above citations do not seem suitable in our case, and our approach is based on refined Agmon estimates for the fiber operators. We will prove the continuity of $(\mathbf{B},\alpha) \mapsto E(\mathbf{B},\mathcal{W}_\alpha)$ on $\mathbb{S}^2 \times (0,2\pi)$, see Theorem 4.5. Let us remark that the continuity is proven even for the degenerate case $\alpha = \pi$. In section 4.4 we improve the result by showing that $(\mathbf{B},\alpha) \mapsto E(\mathbf{B},\mathcal{W}_\alpha)$ is Lipschitz when inequality (1.7) is strict and $\alpha \neq \pi$, that is not surprising because in some sense we are not so far from Kato's perturbation theory in that case since there exists generalized eigenfunctions associated with $E(\mathbf{B},\mathcal{W}_\alpha)$. When (1.7) is an equality, we prove $\frac{1}{3}$-Hölder regularity (see Proposition 4.7). As this stade we do not know whether the $\frac{1}{3}$ exponent is optimal or not. Numerical simulations of $E(\mathbf{B},\mathcal{W}_\alpha)$ as a function of $\alpha \in (0,\pi)$ for a particular $\mathbf{B} \in \mathbb{S}^2$ are provided in Figure 3 and suggest that $E(\mathbf{B},\mathcal{W}_\alpha)$ is not $\mathcal{C}^1$ in general.

The diamagnetic inequality is well known and states that the energy is larger in presence of a magnetic field (see [28] or [48]). A strict diamagnetic inequality has been proved for the Neumann magnetic Laplacian in bounded domains in [20, Chapter 2]. A direct consequence of our analysis is a strict diamagnetic inequality for this problem on an unbounded domain, namely $E(\mathbf{B},\mathcal{W}_\alpha) > 0$ for all non-zero magnetic field $\mathbf{B}$ (see Corollary 3.9).

1.3. **Application of our results to the semi-classical problem.** We come back here to the analysis of the semi-classical magnetic Laplacian on a bounded singular domain Ω. What we call the *local ground state energy* of a point $x \in \overline{\Omega}$ is the bottom of the spectrum of the magnetic Laplacian on the tangent cone to Ω at x with a linear potential associated with the magnetic field frozen at x. It is well known that this local ground state energy is Lipschitz continuous on the regular boundary of Ω (indeed it expresses as a function of the quantity $\sigma(\cdot)$ described above). As said before, the presence of edges in the boundary of Ω leads to the model magnetic Laplacian on wedges that was only described for particular cases and that is systematically studied in this article. The main direct consequence of inequality (1.7) combined with Theorem 4.5 is that the local ground state energy is lower semi-continuous on a domain Ω whose boundary singularities are edges. For a non-vanishing magnetic field $\mathbf{B}$, define $\mathscr{E}(\mathbf{B},\Omega)$ the infimum of the local ground state energy along $\overline{\Omega}$. As a consequence of the lower-semi continuity together with Corollary 3.9, this infimum is reached and $\mathscr{E}(\mathbf{B},\Omega) > 0$. Moreover when inequality (1.7) is strict at x_0 belonging to an edge of Ω, the local ground state energy is discontinuous when coming from faces toward x_0. Using the existence of generalized eigenfunction with exponential decay far from the edge (see Corollary 3.8), standard semi-classical tools bring asymptotics and localization properties for the lowest eigenpairs of the magnetic Laplacian in the semiclassical limit (see [38], [8], [9] and [43]). More precisely the first eigenvalue behaves like $h\mathscr{E}(\mathbf{B},\Omega)+O(h^{5/4})$ (see [40, Section 8] and [43] for particular domains with edges, and [12] for polyhedral domains, on which the results of this article are used). Due to standard Agmon estimates, we also expect that the associated eigenvectors are localized near the minimizers of $\mathscr{E}(\mathbf{B},\Omega)$, that are likely[1], due to (1.7), to be on an edge if Ω has non corners.

Some of our results are key ingredients in order to analyse the asymptotic behavior of the first eigenvalue of $(-ih\nabla-\mathbf{A})^2$ for a non-vanishing magnetic field in a general corner domain Ω. In [11], we show that when Ω belongs to a wide class of corner domains, the first eigenvalue behaves like $h\mathscr{E}(\mathbf{B},\Omega)$ and remainders as a power of h depending on the geometry are provided. The lower semi-continuity near edges is needed when looking for a minimizer of the local ground state energy, and the existence of generalized eigenfunctions for the model Laplacian on the wedge brings quasi-modes for the semi-classical problem. The Lipschitz regularity of the ground state depending on the geometry allows a better estimation of the quasi-mode for the semi-classical problem.

1.4. **State of the art on wedges.** The model operator on infinite wedges has already been explored for particular cases:

In [38], X. B. Pan studies the case of wedges of opening $\frac{\pi}{2}$ and applies his results to the semiclassical problem on a cuboid. In particular he shows that inequality (1.7) is strict if the magnetic field is tangent to a face of the wedge but not to the axis. These results can hardly be extended to the general case.

The case of the magnetic field $\mathbf{B}_0 := (0,0,1)$ tangent to the edge reduces to a magnetic Laplacian on the sector $\mathcal{S}_\alpha$. This case is studied in [8] (see also [27] for $\alpha=\frac{\pi}{2}$): There holds $\mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)=\sigma(0)=\Theta_0$ and it is proven that inequality (1.7) is strict at least for $\alpha\in(0,\frac{\pi}{2}]$. V. Bonnaillie shows in particular that $E(\mathbf{B},\mathcal{W}_\alpha)\sim\frac{\alpha}{\sqrt{3}}$ when $\alpha\to0$ and gives a complete expansion of $E(\mathbf{B},\mathcal{W}_\alpha)$ in power of α.

In [42], a magnetic field tangent to a face of the wedge is considered. In that case inequality (1.7) is proven with $\mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)=\Theta_0$. Moreover it is shown that inequality (1.7) is strict for α small enough but cases of equality are also exhibited.

In [43], the magnetic field is normal to the plane of symmetry of the wedge and it is shown that inequality (1.7) is strict at least for α small enough.

The results of this article cover these particular cases and give a more general approach about the model problem on wedges.

1.5. **Organization of the article.** In Section 2 we reduce the operator $H(\mathbf{A},\mathcal{W}_\alpha)$ to a family of fibers $(\widehat{H}^\tau(\mathbf{A},\mathcal{W}_\alpha))_{\tau\in\mathbb{R}}$ on the sector $\mathcal{S}_\alpha$. In Section 3, we link the problem on the wedge with model operators on half-spaces corresponding to the two faces and we deduce inequality (1.7). In section 4 we prove that $E(\mathbf{B},\mathcal{W}_\alpha)$ is continuous with respect to the geometry defined by $(\mathbf{B},\alpha)\in\mathbb{S}^2\times(0,2\pi)$. We also prove Lipschitz and Hölder regularity depending on whether inequality (1.7) is strict or not. In Section 5 we use a 1d operator to construct quasimodes for

[1]This depends also on the variations of the magnetic field.

α small and we exhibit cases where inequality (1.7) is strict. In Section 6 we give numerical computation of the eigenpairs of the reduced operator on the sector.

2. From the wedge to the sector

2.1. **Reduction to a sector.** Due to the symmetry of the problem (see [40, Proposition 3.14] for the detailed proof) we have the following:

Proposition 2.1. *Let* $\mathbf{B} = (b_1, b_2, b_3)$ *be a constant magnetic field and* $\mathbf{A}$ *an associated potential. The operator* $H(\mathbf{A}, \mathcal{W}_\alpha)$ *is unitary equivalent to* $H(\widetilde{\mathbf{A}}, \mathcal{W}_\alpha)$ *where* $\widetilde{\mathbf{A}}$ *satisfies* $\operatorname{curl} \widetilde{\mathbf{A}} = (|b_1|, |b_2|, |b_3|)$.

Therefore we can restrict ourselves to the case $b_i \geqslant 0$.

We assume that the magnetic potential $\mathbf{A} = (a_1, a_2, a_3)$ satisfies $\operatorname{curl} \mathbf{A} = \mathbf{B}$ and the magnetic Schrödinger operator writes:

$$H(\mathbf{A},\, \mathcal{W}_\alpha) = \sum_{j=1}^{3} (D_{x_j} - a_j)^2$$

with $D_{x_j} = -i\partial_{x_j}$. Due to gauge invariance, the spectrum of $H(\mathbf{A},\, \mathcal{W}_\alpha)$ does not depend on the choice of $\mathbf{A}$ as soon as it satisfies $\operatorname{curl} \mathbf{A} = \mathbf{B}$. Moreover we can choose $\mathbf{A}$ independent of the x_3 variable. The magnetic potential will be chosen explicitly later, see (2.4).

We denote by $\mathfrak{S}(P)$ (respectively $\mathfrak{S}_{\mathrm{ess}}(P)$) the spectrum (respectively the essential spectrum) of an operator P. Due to the invariance by translation in the x_3-variable, there holds $\mathfrak{S}(H(\mathbf{A},\, \mathcal{W}_\alpha)) = \mathfrak{S}_{\mathrm{ess}}(H(\mathbf{A},\, \mathcal{W}_\alpha))$.

2.1.1. *Partial Fourier transform.* Let $\tau \in \mathbb{R}$ be the Fourier variable dual to x_3 and $\mathcal{F}_{x_3}$ the associated Fourier transform. We recall that $\mathbf{A}$ has been chosen independent of the x_3 variable and for $\tau \in \mathbb{R}$ we introduce the operator

$$\widehat{H}^\tau(\mathbf{A}, \mathcal{W}_\alpha) := (D_{x_1} - a_1)^2 + (D_{x_2} - a_2)^2 + (a_3 - \tau)^2 \tag{2.1}$$

acting on $L^2(\mathcal{S}_\alpha)$ with natural Neumann boundary condition. We have the following direct integral decomposition (see [45, Chapter XIII]):

$$\mathcal{F}_{x_3} H(\mathbf{A},\, \mathcal{W}_\alpha) \mathcal{F}_{x_3}^* = \int_{\tau\in\mathbb{R}}^{\oplus} \widehat{H}^\tau(\mathbf{A}, \mathcal{W}_\alpha)\, \mathrm{d}\tau\ . \tag{2.2}$$

Note that this decomposition is quite close to the operators studied in [30, Section 8.2]. The operator $H(\mathbf{A},\, \mathcal{W}_\alpha)$ is a fibered operator (see [21] for a general setting, although our operator does not satisfy fully the definitions of an *analytically fiber operator*) whose fibers are the 2d operators $\widehat{H}^\tau(\mathbf{A}, \mathcal{W}_\alpha)$ with $\tau \in \mathbb{R}$. Let

$$s(\mathbf{B}, \mathcal{S}_\alpha; \tau) := \inf \mathfrak{S}(\widehat{H}^\tau(\mathbf{A}, \mathcal{W}_\alpha))$$

be the bottom of the spectrum of $\widehat{H}^\tau(\mathbf{A}, \mathcal{W}_\alpha)$, also called the *band function.*Thanks to (2.2) we have the following fundamental relation:

$$E(\mathbf{B}, \mathcal{W}_\alpha) = \inf_{\tau\in\mathbb{R}} s(\mathbf{B}, \mathcal{S}_\alpha; \tau)\ . \tag{2.3}$$

As a consequence we are reduced to study the spectrum of a 2d family of Schrödinger operators. We denote by

$$s_{\text{ess}}(\mathbf{B}, \mathcal{S}_\alpha; \tau) := \inf \mathfrak{S}_{\text{ess}}(\widehat{H}^\tau(\mathbf{A}, \mathcal{W}_\alpha))$$

the bottom of the essential spectrum.

2.1.2. *Description of the reduced operator.* We write

$$\mathbf{B} = \mathbf{B}^\perp + \mathbf{B}^\parallel$$

where $\mathbf{B}^\perp = (b_1, b_2, 0)$ and $\mathbf{B}^\parallel = (0, 0, b_3)$. We take for the magnetic potential

$$\mathbf{A}(x_1, x_2, x_3) = (\mathbf{A}^\parallel(x_1, x_2), a^\perp(x_1, x_2)) \tag{2.4}$$

with $\mathbf{A}^\parallel(x_1, x_2) := (0, b_3 x_1)$ and $a^\perp(x_1, x_2) = x_2 b_1 - x_1 b_2$. The magnetic potentiel $\mathbf{A}$ is linear, does not depend on x_3 and satisfies $\operatorname{curl} \mathbf{A} = \mathbf{B}$. We introduce the reduced electric potential on the sector:

$$V^\tau_{\mathbf{B}^\perp}(x_1, x_2) := (x_2 b_1 - x_1 b_2 - \tau)^2 .$$

We have

$$\widehat{H}_\tau(\mathbf{A}, \mathcal{W}_\alpha) = H(\mathbf{A}^\parallel, \mathcal{S}_\alpha) + V^\tau_{\mathbf{B}^\perp} . \tag{2.5}$$

The quadratic form of $H(\mathbf{A}^\parallel, \mathcal{S}_\alpha) + V^\tau_{\mathbf{B}^\perp}$ is

$$\mathcal{Q}^\tau_{\mathbf{B},\alpha}(u) := \int_{\mathcal{S}_\alpha} |(-i\nabla - \mathbf{A}^\parallel)u|^2 + V^\tau_{\mathbf{B}^\perp}|u|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2$$

defined on the form domain

$$\operatorname{Dom}(\mathcal{Q}^\tau_{\mathbf{B},\alpha}) = \{u \in L^2(\mathcal{S}_\alpha),\ (-i\nabla - \mathbf{A}^\parallel)u \in L^2(\mathcal{S}_\alpha),\ |x_2 b_1 - x_1 b_2 - \tau| u \in L^2(\mathcal{S}_\alpha)\} . \tag{2.6}$$

The form domain coincides with:

$$\{u \in L^2(\mathcal{S}_\alpha),\ (-i\nabla - \mathbf{A}^\parallel)u \in L^2(\mathcal{S}_\alpha),\ |x_2 b_1 - x_1 b_2| u \in L^2(\mathcal{S}_\alpha)\} ,$$

therefore it does not depend on τ. Kato's perturbation theory (see [29]) provides the following:

Proposition 2.2. *The function $\tau \mapsto s(\mathbf{B}, \mathcal{S}_\alpha; \tau)$ is continuous on $\mathbb{R}$.*

2.2. **Model problems on regular domain.** We describe here the case $\alpha = \pi$ where $\mathcal{W}_\pi$ is a half-space. The operator $H(\mathbf{A}^\parallel, \mathcal{S}_\pi) + V^\tau_{\mathbf{B}^\perp}$ can be analyzed using known results about regular domain. We have $E(\mathbf{B}, \mathcal{W}_\pi) = \sigma(\theta)$ (see Subsection 1.2 and [23]) where $\theta \in [0, \frac{\pi}{2}]$ is the angle between the magnetic field and the boundary. We recall that we have $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\pi) = 1$.

When $\theta \neq 0$, $H(\mathbf{A}^\parallel, \mathcal{S}_\pi) + V^\tau_{\mathbf{B}^\perp}$ is unitary equivalent to $H(\mathbf{A}^\parallel, \mathcal{S}_\pi) + V^0_{\mathbf{B}^\perp}$ and $s_{\text{ess}}(\mathbf{B}, \mathcal{S}_\pi; 0) = 1$ ([23, Proposition 3.4]). There holds $s(\mathbf{B}, \mathcal{S}_\pi; 0) = \sigma(\theta) \leqslant 1$. If $\theta \neq \frac{\pi}{2}$, $\sigma(\theta) < 1$ and therefore the operator $H(\mathbf{A}^\parallel, \mathcal{S}_\pi) + V^0_{\mathbf{B}^\perp}$ has an eigenfunction associated with $\sigma(\theta)$ with exponential decay (see [13]).

When $\theta = 0$, there holds $s_{\text{ess}}(\mathbf{B}, \mathcal{S}_\pi; \tau) = s(\mathbf{B}, \mathcal{S}_\pi; \tau)$. A partial Fourier transform can be performed and shows that $\inf_{\tau \in \mathbb{R}} s(\mathbf{B}, \mathcal{S}_\pi; \tau) = \Theta_0$.

In Subsection 2.3 and Section 3 we will focus on $\alpha \in (0, \pi) \cup (\pi, 2\pi)$. Most of the results can be compared and extended to $\alpha = \pi$ using the results recalled above.

2.3. **Link between the geometry and the essential spectrum of the reduced problem.** In this section we give the essential spectrum of the operator $H(\mathbf{A}^{\parallel}, \mathcal{S}_\alpha) + V^{\tau}_{\mathbf{B}^{\perp}}$ depending on the geometry. Let $\Upsilon := (V^{\tau}_{\mathbf{B}^{\perp}})^{-1}(\{0\})$ be the line where the electric potential vanishes. Let us notice that $V^{\tau}_{\mathbf{B}^{\perp}}(x)$ is the square of the distance from x to Υ. Let (γ, θ) be the spherical coordinates of the magnetic field where γ is the angle between the magnetic field and the x_3-axis and θ is the angle between the projection (b_1, b_2) and the x_2-axis:

$$\mathbf{B} = (\sin\gamma\sin\theta, \sin\gamma\cos\theta, \cos\gamma) \, .$$

Due to symmetries we restrict ourselves to $(\gamma, \theta) \in [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$. We will use the following terminology:

- The magnetic field is outgoing if $\alpha \in (0, \pi)$ and $\theta \in [0, \frac{\pi-\alpha}{2})$.
- The magnetic field is tangent if either $\gamma = 0$ or $\theta = \frac{|\pi-\alpha|}{2}$.
- The magnetic field is ingoing in the other cases.

The outgoing case corresponds to a magnetic field pointing outward the wedge (this can happen only if the wedge is convex). The tangent case corresponds to a magnetic field tangent to a face of the wedge and has already been explored for convex wedges in [42]. The ingoing case corresponds to a magnetic field pointing inward the wedge, in that case the intersection between Υ and $\mathcal{S}_\alpha$ is always unbounded. The essential spectrum of $H(\mathbf{A}^{\parallel}, \mathcal{S}_\alpha) + V^{\tau}_{\mathbf{B}^{\perp}}$ depends on the situation as described below:

Proposition 2.3. *Let $\alpha \in (0, \pi)$ and $\mathbf{B} \in \mathbb{S}^2$ be an outgoing magnetic field. Then for all $\tau \in \mathbb{R}$ the operator $H(\mathbf{A}^{\parallel}, \mathcal{S}_\alpha) + V^{\tau}_{\mathbf{B}^{\perp}}$ has compact resolvent.*

Proof. We remark that

$$\forall \tau \in \mathbb{R}, \qquad \lim_{\substack{|(x_1,x_2)| \to +\infty \\ (x_1,x_2) \in \mathcal{S}_\alpha}} V^{\tau}_{\mathbf{B}^{\perp}}(x_1, x_2) = +\infty \, .$$

This implies that the injection from the form domain (2.6) into $L^2(\mathcal{S}_\alpha)$ is compact, see for example [45]. We deduce that the operator $H(\mathbf{A}^{\parallel}, \mathcal{S}_\alpha) + V^{\tau}_{\mathbf{B}^{\perp}}$ has compact resolvent. ☐

The following proposition shows that the essential spectrum is much more different when the magnetic field is ingoing:

Proposition 2.4. *Let $\alpha \in (0, \pi) \cup (\pi, 2\pi)$ and $\mathbf{B} \in \mathbb{S}^2$ be an ingoing magnetic field. Then*

$$\forall \tau \in \mathbb{R}, \quad s_{\text{ess}}(\mathbf{B}, \mathcal{S}_\alpha; \tau) = 1 \, .$$

When $\alpha \in (0, \pi)$, the detailed proof can be found in [40, Subsection 4.2.2]. The proof for $\alpha \in (\pi, 2\pi)$ is rigorously the same. The idea is to construct a Weyl's quasimode for $\mathcal{Q}^{\tau}_{\mathbf{B},\alpha}$ far from the origin and near the line Υ using the operator $H(\mathbf{A}^{\parallel}, \mathbb{R}^2) + V^{\tau}_{\mathbf{B}^{\perp}}$ whose first eigenvalue is 1. Persson's lemma (see [39]) provides the result.

In the tangent case, the essential spectrum depends on the parameters and can be expressed using the first eigenvalue of the classical 1d de Gennes operator (see the proof below). The bottom of the essential spectrum is given explicitly in (2.7) however we will only need the following:

Proposition 2.5. *Let $\alpha \in (0,\pi) \cup (\pi, 2\pi)$ and $\mathbf{B} \in \mathbb{S}^2$ be a magnetic field tangent to $\mathcal{W}_\alpha$. Then we have*
$$\inf_{\tau \in \mathbb{R}} s_{\,\mathrm{ess}}(\mathbf{B}, \mathcal{S}_\alpha; \tau) = \Theta_0 \,.$$

Proof. We introduce the first eigenvalue $\mu(\xi)$ of the 1d de Gennes operator
$$-\partial_t^2 + (t-\xi)^2$$
defined on the half-line $\{t > 0\}$ with a Neumann boundary condition. This classical spectral quantity has already been investigated, see [47, 6, 18]. In particular $\mu(\xi)$ reaches a unique minimum $\Theta_0 \approx 0.59$ for $\xi_0 = \sqrt{\Theta_0}$. We recall the result from [42, Proposition 3.6]:
$$s_{\,\mathrm{ess}}(\mathbf{B}, \mathcal{S}_\alpha; \tau) = \inf_{\xi \in \mathbb{R}} \left(\mu(\xi\cos\gamma + \tau\sin\gamma) + (\xi\sin\gamma - \tau\cos\gamma)^2\right) \,. \tag{2.7}$$
where $\gamma \in [0, \frac{\pi}{2}]$ is the angle between the magnetic field and the axis of the wedge. Note that the proof of this relation is done in [42] for $\alpha \in (0,\pi)$ and the extension to $\alpha \in (\pi, 2\pi)$ does not need any additional work. We deduce from (2.7) that
$$\forall \tau \in \mathbb{R}, \quad s_{\,\mathrm{ess}}(\mathbf{B}, \mathcal{S}_\alpha; \tau) \geqslant \Theta_0 \,. \tag{2.8}$$
Choosing $\xi = \xi_0 \cos\gamma$ in the r.h.s. of (2.7) and $\tau = \xi_0 \sin\gamma$ we get $s_{\,\mathrm{ess}}(\mathbf{B}, \mathcal{S}_\alpha, \xi_0\sin\gamma) = \mu(\xi_0) = \Theta_0$ and the proposition is proven. □

Remark 2.6. We have $\sigma(0) = \Theta_0$ where the function σ is defined in Subsection 1.2.

Since $s(\mathbf{B}, \mathcal{S}_\alpha; \tau) \leqslant s_{\,\mathrm{ess}}(\mathbf{B}, \mathcal{S}_\alpha; \tau)$, the relation (2.3) provides for a tangent magnetic field:
$$\forall \alpha \in (0, 2\pi)\backslash\pi, \quad E(\mathbf{B}, \mathcal{W}_\alpha) \leqslant \Theta_0 \,. \tag{2.9}$$
Therefore we have proven inequality (1.7) for a tangent magnetic field.

3. Link with problems on half-planes

In this section we will investigate the link between the model operator on a wedge of opening $\alpha \in (0,\pi) \cup (\pi, 2\pi)$ and the model operators on the half-spaces Π_α^+, Π_α^- and the space $\mathbb{R}^3$ (see Subsection 1.2). These domains are the tangent substructure of $\mathcal{W}_\alpha$. We recall that $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha)$ is the lowest energy of the magnetic Laplacian $(-i\nabla - \mathbf{A})^2$ acting on these tangent substructures and is given by
$$\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) = \sigma(\min\{\theta^+, \theta^-\})$$
where $\theta^\pm$ is the angle between $\mathbf{B}$ and $\Pi_\alpha^\pm$ and $\sigma(\cdot)$ is defined in Subsection 1.2. In this section we prove inequality (1.7). Moreover when this inequality is strict we show that the band function $\tau \mapsto s(\mathbf{B}, \mathcal{S}_\alpha; \tau)$ reaches its infimum and that this infimum is a discrete eigenvalue for the reduced operator on the sector. Let us remark that these questions were investigated in [38] and [42] for particular cases.

We denote by $\mathcal{H}_\alpha^+$ and $\mathcal{H}_\alpha^-$ the half-planes such that $\Pi_\alpha^+ = \mathbb{R} \times \mathcal{H}_\alpha^+$ and $\Pi_\alpha^- = \mathbb{R} \times \mathcal{H}_\alpha^-$. Let $H(\mathbf{A}^\parallel, \mathcal{H}_\alpha^+) + V_{\mathbf{B}^\perp}^\tau$ be the reduced operator defined on $\mathcal{H}_\alpha^+$ with a Neumann boundary condition. When $\mathbf{B}$ is not tangent to Π_α^+ we deduce from Subsection 2.2:
$$\forall \tau \in \mathbb{R}, \quad \inf \mathfrak{S}(H(\mathbf{A}^\parallel, \mathcal{H}_\alpha^+) + V_{\mathbf{B}^\perp}^\tau) = \sigma(\theta^+) \tag{3.1}$$

Similarly when the magnetic field is not tangent to Π_α^- we have:

$$\forall \tau \in \mathbb{R}, \quad \inf \mathfrak{S}(H(\mathbf{A}^{\|}, \mathcal{H}_\alpha^-) + V_{\mathbf{B}^\perp}^\tau) = \sigma(\theta^-) \tag{3.2}$$

3.1. **Limits for large Fourier parameter.** In this section we investigate the behavior of $s(\mathbf{B}, \mathcal{S}_\alpha; \tau)$ when the Fourier parameter τ goes to $\pm\infty$. We introduce the quantity

$$s^\infty(\mathbf{B}, \mathcal{S}_\alpha) := \min \left\{ \liminf_{\tau\to-\infty} s(\mathbf{B}, \mathcal{S}_\alpha; \tau), \liminf_{\tau\to+\infty} s(\mathbf{B}, \mathcal{S}_\alpha; \tau) \right\} . \tag{3.3}$$

In the tangent case, we recall the results from [42, Section 4]:

Proposition 3.1. *Let $\alpha \in (0, \pi) \cup (\pi, 2\pi)$ and let $\mathbf{B} \in \mathbb{S}^2$ be a magnetic field tangent to a face of the wedge $\mathcal{W}_\alpha$. Then we have*

$$s^\infty(\mathbf{B}, \mathcal{S}_\alpha) = \sigma(\max(\theta^-, \theta^+)) .$$

Note that in [42], this result is proved only for $\alpha \in (0, \pi)$. The proof of [42, Proposition 4.1] is mimicked to the case $\alpha \in (\pi, 2\pi)$.

We recall the useful *IMS localization formula* (see [16, Theorem 3.2] and also [49]):

Lemma 3.2. *Let (χ_j) be a finite regular partition of the unity satisfying $\sum \chi_j^2 = 1$. We have for $u \in \mathrm{Dom}(\mathcal{Q}_{\mathbf{B},\alpha}^\tau)$:*

$$\mathcal{Q}_{\mathbf{B},\alpha}^\tau(u) = \sum_j \mathcal{Q}_{\mathbf{B},\alpha}^\tau(\chi_j u) - \sum_j \|\nabla \chi_j u\|_{L^2}^2 .$$

The following lemma gives a lower bound on the energy of a function supported far from the corner of the sector. This lemma will also be useful in Section 4. We denote by $B(0, R)$ the ball centered at the origin of radius $R > 0$ and $\complement B(0, R)$ its complement.

Lemma 3.3. *There exist $C_1 > 0$ and $R_0 > 0$ such that for all $\alpha \in (0, \pi) \cup (\pi, 2\pi)$ and for all $\mathbf{B} \in \mathbb{S}^2$, for all $R \geqslant R_0$, for all $\tau \in \mathbb{R}$, for all $u \in \mathrm{Dom}(\mathcal{Q}_{\mathbf{B},\alpha}^\tau)$ such that $\mathrm{Supp}(u) \subset \complement\overline{B(0, R)}$:*

$$\mathcal{Q}_{\mathbf{B},\alpha}^\tau(u) \geqslant \left(\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) - \frac{C_1}{\alpha^2 R^2} \right) \|u\|_{L^2}^2 .$$

Proof. Let $(\chi_j)_{j=1,2,3}$ be a partition of unity satisfying $\chi_j \in C_0^\infty([-\frac{1}{2}, \frac{1}{2}], [0, 1])$, $\mathrm{Supp}(\chi_j) \subset [\frac{j-3}{4}, \frac{j-1}{4}]$ and $\sum_j \chi_j^2 = 1$. We defined the cut-off functions $\chi_{j,\alpha}^{\mathrm{pol}}(r, \psi) := \chi_j(\frac{\psi}{\alpha})$ where $(r, \psi) \in \mathbb{R}_+ \times (-\frac{\alpha}{2}, \frac{\alpha}{2})$ are the polar coordinates. We denote by $\chi_{j,\alpha}$ the associated functions in cartesian coordinates. Since the $\chi_{j,\alpha}$ do not depend on r, there exists $C_1 > 0$ such that

$$\forall \alpha \in (0, 2\pi), \forall R > 0, \quad \forall (x_1, x_2) \in \complement B(0, R), \quad \sum_{j=1}^3 |\nabla \chi_{j,\alpha}(x_1, x_2)|^2 \leqslant \frac{C_1}{R^2 \alpha^2} .$$

Let $u \in \mathrm{Dom}\, \mathcal{Q}_{\mathbf{B},\alpha}^\tau$ such that $\mathrm{Supp}(u) \subset \complement B(0, R)$. The IMS formula (see Lemma 3.2) provides

$$\mathcal{Q}_{\mathbf{B},\alpha}^\tau(u) \geqslant \sum_{j=1}^3 \mathcal{Q}_{\mathbf{B},\alpha}^\tau(\chi_{j,\alpha} u) - \frac{C_1}{\alpha^2 R^2} \|u\|_{L^2}^2 . \tag{3.4}$$

Moreover $\chi_1 u$ and $\chi_3 u$ are extended to functions of $L^2(\mathcal{H}_\alpha^+)$ and $L^2(\mathcal{H}_\alpha^-)$ with the suitable Neumann boundary conditions by setting $\chi_j u = 0$ outside $\mathrm{Supp}(\chi_j)$. We deduce from (3.1) and the min-max principle that $\mathcal{Q}^\tau_{\mathbf{B},\alpha}(\chi_{1,\alpha}u) \geqslant \sigma(\theta^+)\|\chi_{1,\alpha}u\|^2_{L^2}$. Similarly we prove $\mathcal{Q}^\tau_{\mathbf{B},\alpha}(\chi_{3,\alpha}u) \geqslant \sigma(\theta^-)\|\chi_{3,\alpha}u\|^2_{L^2}$. The function $\chi_{2,\alpha}u$ is extended in the same way to a function of $\mathbb{R}^2$. It is elementary that

$$\forall \tau \in \mathbb{R}, \quad \inf \mathfrak{S}(H(\mathbf{A}^{\|}, \mathbb{R}^2) + V^\tau_{\mathbf{B}^\perp}) = E(\mathbf{B}, \mathbb{R}^3) = 1 ,$$

therefore $\mathcal{Q}^\tau_{\mathbf{B},\alpha}(\chi_{2,\alpha}u) \geqslant \|\chi_{2,\alpha}u\|^2_{L^2}$. We conclude with (3.4) and the definition of $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha)$ (see (1.5)). ☐

Proposition 3.4. *Let $\alpha \in (0, \pi) \cup (\pi, 2\pi)$ and let $\mathbf{B} \in \mathbb{S}^2$ be a magnetic field which is not tangent to a face of the wedge $\mathcal{W}_\alpha$. We have*

$$s^\infty(\mathbf{B}, \mathcal{S}_\alpha) = \mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) . \tag{3.5}$$

Remark 3.5. The relation (3.5) is not true when the magnetic field is tangent to a face of the wedge, see Proposition 3.1 and (1.6).

Proof. LOWER BOUND: Let (χ_1, χ_2) be two cut-off functions in $\mathcal{C}^\infty(\mathbb{R}_+, [0, 1])$ satisfying $\chi_1^2 + \chi_2^2 = 1$, $\chi_1(r) = 1$ if $r \in (0, \frac{1}{2})$ and $\chi_1(r) = 0$ if $r \in (\frac{3}{4}, +\infty)$. For $\tau \in \mathbb{R}^*$ we define the cut-off functions $\chi_{j,\tau}(x_1, x_2) := \chi_j(\frac{r}{|\tau|})$ with $r = \sqrt{x_1^2 + x_2^2}$. We have

$$\exists C > 0, \forall \tau \in \mathbb{R}^*, \quad \forall (x_1, x_2) \in \mathbb{R}^2, \quad \sum_{j=1}^{2} |\nabla \chi_{j,\tau}|^2 \leqslant \frac{C}{\tau^2} .$$

For $u \in \mathrm{Dom}(\mathcal{Q}^\tau_{\mathbf{B},\alpha})$, the IMS formula (see Lemma 3.2) provides

$$\mathcal{Q}^\tau_{\mathbf{B},\alpha}(u) \geqslant \sum_{j=1}^{2} \mathcal{Q}^\tau_{\mathbf{B},\alpha}(\chi_{j,\tau}u) - \frac{C}{\tau^2}\|u\|^2_{L^2} . \tag{3.6}$$

Since $\mathrm{Supp}(\chi_{1,\tau}) \subset B(0, \frac{3}{4}\tau)$, we have $\mathrm{dist}(\Upsilon, \mathrm{Supp}(\chi_{1,\tau})) \geqslant \frac{\tau}{4}$ and therefore we have

$$\forall (x_1, x_2) \in \mathrm{Supp}(\chi_{1,\tau}), \quad V^\tau_{\mathbf{B}^\perp}(x_1, x_2) \geqslant \tfrac{1}{16}\tau^2 .$$

We deduce that for all $\tau \neq 0$:

$$\mathcal{Q}^\tau_{\mathbf{B},\alpha}(\chi_{1,\tau}u) \geqslant \tfrac{\tau^2}{16}\|\chi_{1,\tau}u\|^2_{L^2} . \tag{3.7}$$

On the other part Lemma 3.3 provides a constant $C_1 > 0$ such that for all $u \in \mathrm{Dom}(\mathcal{Q}^\tau_{\mathbf{B},\alpha})$ we have:

$$\forall \tau \in \mathbb{R}^*, \quad \mathcal{Q}^\tau_{\mathbf{B},\alpha}(\chi_{2,\tau}u) \geqslant \left(\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) - \frac{C_1}{\alpha^2\tau^2}\right)\|\chi_{2,\tau}u\|^2_{L^2} .$$

We deduce by combining this with (3.6) and (3.7) that

$$\mathcal{Q}^\tau_{\mathbf{B},\alpha}(u) \geqslant \min\left\{\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) - \frac{C_1}{\alpha^2\tau^2}, \frac{\tau^2}{16}\right\}\|u\|^2_{L^2} - \frac{C}{\tau^2}\|u\|^2_{L^2} .$$

We deduce from the min-max principle that there exists $\tau_0 > 0$ such that for all τ satisfying $|\tau| > \tau_0$:

$$s(\mathbf{B}, \mathcal{S}_\alpha; \tau) \geqslant \mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) - \frac{C_1}{\alpha^2\tau^2} - \frac{C}{\tau^2}$$

and therefore

$$s^{\infty}(\mathbf{B},\mathcal{S}_\alpha) \geqslant \mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)\,.$$

UPPER BOUND: We suppose that $\theta^+ \leqslant \theta^-$, the other case being symmetric. We have in that case $\mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha) = \sigma(\theta^+)$. Since we have assumed that we are not in the tangent case, we have $0 < \theta^+$. Let $\epsilon > 0$, we deduce from (3.1) that there exists $u_\epsilon \in \mathcal{C}_0^\infty(\overline{\mathcal{H}_\alpha^+})$ such that

$$\big\langle \big(H(\mathbf{A}^{\parallel},\mathcal{H}_\alpha^+) + V^0_{\mathbf{B}^\perp}\big)\, u_\epsilon, u_\epsilon \big\rangle_{L^2(\mathcal{H}_\alpha^+)} = \sigma(\theta^+) + \epsilon\,. \tag{3.8}$$

We use u_ϵ to construct a quasimode of energy $\sigma(\theta^+)+\epsilon$. Let $\mathbf{t}^+ := (\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2})$ be a vector tangent to the boundary of $\mathcal{H}_\alpha^+$. For $x=(x_1,x_2)$, we define the test-function:

$$u_{\epsilon,\tau}(x) := e^{i\tau x\wedge \mathbf{A}^{\parallel}(\mathbf{t}^+)} u_\epsilon(x-\tau\mathbf{t}^+)\,.$$

We have $\mathrm{Supp}(u_{\epsilon,\tau}) = \mathrm{Supp}(u_\epsilon) + \tau\mathbf{t}^+$. Since $\mathbf{t}^+$ is pointing outward the corner of $\mathcal{S}_\alpha$ along the upper boundary, there exists $\tau_0 > 0$ such that for all $\tau > \tau_0$ we have $\mathrm{Supp}(u_{\epsilon,\tau}) \subset \overline{\mathcal{S}_\alpha}$ and $\mathrm{Supp}(u_{\epsilon,\tau}) \cap \partial\Pi_\alpha^- = \varnothing$. Therefore $u_{\epsilon,\tau} \in \mathrm{Dom}(\mathcal{Q}^\tau_{\mathbf{B},\alpha})$. Elementary computations (see the geometrical meaning of $V^\tau_{\mathbf{B}^\perp}(x)$ in Subsection 2.3) provides $V^\tau_{\mathbf{B}^\perp}(x-\tau\mathbf{t}^+) = V^0_{\mathbf{B}^\perp}(x)$. Due to gauge invariance we get

$$\big\langle \big(H(\mathbf{A}^{\parallel},\mathcal{S}_\alpha) + V^\tau_{\mathbf{B}^\perp}\big)\, u_{\epsilon,\tau}, u_{\epsilon,\tau} \big\rangle_{L^2(\mathcal{S}_\alpha)} = \big\langle \big(H(\mathbf{A}^{\parallel},\mathcal{H}_\alpha^+) + V^0_{\mathbf{B}^\perp}\big)\, u_\epsilon, u_\epsilon \big\rangle_{L^2(\mathcal{H}_\alpha^+)}\,.$$

We deduce from (3.8) and from the min-max principle that

$$\forall \epsilon > 0, \exists \tau_0 > 0, \forall \tau > \tau_0, \quad s(\mathbf{B},\mathcal{S}_\alpha;\tau) \leqslant \sigma(\theta^+) + \epsilon$$

and therefore $\liminf_{\tau\to+\infty} s(\mathbf{B},\mathcal{S}_\alpha;\tau) \leqslant \sigma(\theta^+)$. Remark that in this proof we have taken $\tau \to +\infty$ in order to construct a test-function of energy close to $\sigma(\theta^+)$. When $\theta^- \leqslant \theta^+$, the proof is the same but we use $\tau \to -\infty$. □

3.2. Comparison with the spectral quantities coming from the regular case.

Theorem 3.6. *Let $\alpha \in (0,\pi)\cup(\pi,2\pi)$ and $\mathbf{B} \in \mathbb{S}^2$, we have*

$$E(\mathbf{B},\mathcal{W}_\alpha) \leqslant \mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)\,. \tag{3.9}$$

Moreover if $E(\mathbf{B},\mathcal{W}_\alpha) < \mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)$ then the band function $\tau \mapsto s(\mathbf{B},\mathcal{S}_\alpha;\tau)$ reaches its infimum. We denote by $\tau^{\mathrm{c}} \in \mathbb{R}$ a critical point such that

$$s(\mathbf{B},\mathcal{S}_\alpha;\tau^{\mathrm{c}}) = E(\mathbf{B},\mathcal{W}_\alpha)\,.$$

Then there exists an eigenfunction with exponential decay for the operator $H(\mathbf{A}^{\parallel},\mathcal{S}_\alpha) + V^{\tau^{\mathrm{c}}}_{\mathbf{B}^\perp}$ associated with the value $E(\mathbf{B},\mathcal{W}_\alpha)$.

Remark 3.7. Note that in the tangent case the band function $\tau \mapsto s(\mathbf{B},\mathcal{S}_\alpha;\tau)$ always reaches its infimum.

Proof. Tangent case: We have $\mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha) = \Theta_0$ and (3.9) is already proven (see (2.9)). Since the function $\tau \mapsto s(\mathbf{B},\mathcal{S}_\alpha;\tau)$ is continuous, we deduce from Proposition 3.1 and (2.3) that the band function $\tau \mapsto s(\mathbf{B},\mathcal{S}_\alpha;\tau)$ reaches its infimum. Let τ^{c} be a minimizer of $s(\mathbf{B},\mathcal{S}_\alpha;\tau)$. Assume that $E(\mathbf{B},\mathcal{W}_\alpha) < \mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)$. Since $s_{\,\mathrm{ess}}(\mathbf{B},\mathcal{S}_\alpha;\tau^{\mathrm{c}}) \geqslant \Theta_0$ (see Proposition 2.5), $s(\mathbf{B},\mathcal{S}_\alpha;\tau^{\mathrm{c}})$ is a discrete eigenvalue of the operator $H(\mathbf{A}^{\parallel},\mathcal{S}_\alpha) + V^{\tau^{\mathrm{c}}}_{\mathbf{B}^\perp}$.

Non tangent case: We deduce (3.9) from Proposition 3.4 and (2.3). Assume that $E(\mathbf{B},\mathcal{W}_\alpha) < \mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)$. Since the function $\tau \mapsto s(\mathbf{B},\mathcal{S}_\alpha;\tau)$ is continuous, Proposition 3.4 and (2.3) imply that the band function $\tau \mapsto s(\mathbf{B},\mathcal{S}_\alpha;\tau)$ reaches its infimum. We denote by τ^{c} a Fourier parameter such that $E(\mathbf{B},\mathcal{W}_\alpha) = s(\mathbf{B},\mathcal{S}_\alpha;\tau^{\mathrm{c}})$. The bottom of the essential spectrum of $H(\mathbf{A}^{\parallel},\mathcal{S}_\alpha) + V_{\mathbf{B}^\perp}^{\tau^{\mathrm{c}}}$ is either $+\infty$ (outgoing case) or 1 (ingoing case), see Subsection 2.3. Since $\mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha) < 1$ we deduce that $E(\mathbf{B},\mathcal{W}_\alpha)$ is a discrete eigenvalue of the operator $H(\mathbf{A}^{\parallel},\mathcal{S}_\alpha) + V_{\mathbf{B}^\perp}^{\tau^{\mathrm{c}}}$.

In both cases we denote by $u_{\tau^{\mathrm{c}}}$ an eigenfunction associated with $E(\mathbf{B},\mathcal{W}_\alpha)$ for the operator $H(\mathbf{A}^{\parallel},\mathcal{S}_\alpha) + V_{\mathbf{B}^\perp}^{\tau^{\mathrm{c}}}$. The fact that $u_{\tau^{\mathrm{c}}}$ has exponential decay is classical (see [1]) and we will give precise informations about the decay rate of the eigenfunctions in Proposition 4.2. $\square$

Several particular cases where $E(\mathbf{B},\mathcal{W}_\alpha) < \mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)$ can be found in literature (see [38], [8] or [42]). Theorem 5.4 below gives geometrical conditions for this inequality to be satisfied. Let us also note that in [42, Section 5], it is proved that $E(\mathbf{B},\mathcal{W}_\alpha) = \mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)$ for a magnetic field tangent to a face, normal to the edge with an opening angle larger that $\frac{\pi}{2}$.

We now show that when inequality (1.7) is strict, there exists a generalized eigenfunction (in some sense we will define below) for $H(\mathbf{A},\mathcal{W}_\alpha)$ associated with the ground state energy $E(\mathbf{B},\mathcal{W}_\alpha)$. This generalized eigenfunction is localized near the edge and can be used to construct quasimodes for the semiclassical magnetic Laplacian on a bounded domain with edges (see [12]).

We denote by $L^2_{\mathrm{loc}}(\overline{\mathcal{W}_\alpha})$ (respectively $H^1_{\mathrm{loc}}(\overline{\mathcal{W}_\alpha})$) the set of the functions u which are in $L^2(\mathring{K})$ (respectively $H^1(\mathring{K})$) for all compact K included in $\overline{\mathcal{W}_\alpha}$ where $\mathring{K}$ denotes the interior of K.

We introduce the set of the functions which are *locally* in the domain of $H(\mathbf{A},\mathcal{W}_\alpha)$:

$$\mathrm{Dom}_{\mathrm{loc}}(H(\mathbf{A},\mathcal{W}_\alpha)) := \\ \{u \in H^1_{\mathrm{loc}}(\overline{\mathcal{W}_\alpha}),\ (-i\nabla - \mathbf{A})^2 u \in L^2_{\mathrm{loc}}(\overline{\mathcal{W}_\alpha}),\ (-i\nabla - \mathbf{A})u\cdot\mathbf{n} = 0 \ \text{ on } \ \partial\mathcal{W}_\alpha\},$$

where $\mathbf{n}$ is the outward normal of the boundary $\partial\mathcal{W}_\alpha$ of the wedge.

Corollary 3.8. *Let $\alpha \in (0,\pi)\cup(\pi,2\pi)$ and $\mathbf{B} \in \mathbb{S}^2$. Assume $E(\mathbf{B},\mathcal{W}_\alpha) < \mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)$. Then there exists a non-zero function $\psi \in \mathrm{Dom}_{\mathrm{loc}}(H(\mathbf{A},\mathcal{W}_\alpha))$ satisfying*

$$\begin{cases} (-i\nabla - \mathbf{A})^2\psi = E(\mathbf{B},\mathcal{W}_\alpha)\psi & \text{in} \quad \mathcal{W}_\alpha \\ (-i\nabla - \mathbf{A})\psi\cdot\mathbf{n} = 0 & \text{on} \quad \partial\mathcal{W}_\alpha\,. \end{cases}$$

Moreover ψ has exponential decay in the (x_1,x_2) variables.

Proof. Let τ^{c} be a minimizer of $\tau \mapsto s(\mathbf{B},\mathcal{S}_\alpha;\tau)$ given by Theorem 3.6. Let $u_{\tau^{\mathrm{c}}}$ be an eigenfunction of $H(\mathbf{A}^{\parallel},\mathcal{S}_\alpha)+V_{\mathbf{B}^\perp}^{\tau^{\mathrm{c}}}$ associated with $E(\mathbf{B},\mathcal{W}_\alpha)$. It has exponential decay and satisfies the boundary condition $(-i\nabla - \mathbf{A}^{\parallel})u_{\tau^{\mathrm{c}}}\cdot\underline{\mathbf{n}} = 0$ where $\underline{\mathbf{n}}$ is the outward normal to the boundary of $\mathcal{S}_\alpha$. Let

$$\psi(x_1,x_2,x_3) := e^{i\tau^{\mathrm{c}}x_3}u_{\tau^{\mathrm{c}}}(x_1,x_2)\,. \tag{3.10}$$

We clearly have $\psi \in \mathrm{Dom}_{\mathrm{loc}}(H(\mathbf{A}, \mathcal{W}_\alpha))$. Moreover writing $\mathbf{A} = (\mathbf{A}^{\|}, x_2 b_1 - x_1 b_2)$ we get

$$(-i\nabla - \mathbf{A})^2\psi = \left((-i\nabla_{x_1,x_2} - \mathbf{A}^{\|})^2 u_{\tau^{\mathrm{c}}} + (\tau^{\mathrm{c}} - x_2 b_1 + x_1 b_2)^2 u_{\tau^{\mathrm{c}}}\right) e^{i\tau^{\mathrm{c}} x_3} = E(\mathbf{B}, \mathcal{W}_\alpha)\psi\,.$$

Therefore ψ satisfies the conditions of the corollary. □

We say that the function ψ is a generalized eigenfunction of $H(\mathbf{A}, \mathcal{W}_\alpha)$. Since it has the form (3.10), we say it is *admissible* and we shall use it to construct quasimode for the operator $(-ih\nabla - \mathbf{A})^2$ on Ω when Ω has an edge (see [12]). This form is linked to the notion of L^∞ *spectral pair*, see for example [2, Section 2.4].

We also deduce from Theorem 3.6 the following strict diamagnetic inequality:

Corollary 3.9. *Let* $(\mathbf{B}, \alpha) \in \mathbb{S}^2 \times (0, 2\pi)$. *Then we have* $E(\mathbf{B}, \mathcal{W}_\alpha) > 0$.

Proof. Assume $E(\mathbf{B}, \mathcal{W}_\alpha) = 0$, then using (1.6) there holds $E(\mathbf{B}, \mathcal{W}_\alpha) < \mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha)$ and we use Theorem 3.6: there exists $\tau^{\mathrm{c}} \in \mathbb{R}$ and $u_{\tau^{\mathrm{c}}}$ a non-zero eigenfunction for the operator $H(\mathbf{A}^{\|}, \mathcal{S}_\alpha) + V_{\mathbf{B}^\perp}^{\tau^{\mathrm{c}}}$ associated with 0. Looking at the associated rayleigh quotient we get

$$\int_{\mathcal{S}_\alpha} |(-i\nabla - \mathbf{A}^{\|})u_{\tau^{\mathrm{c}}}|^2 + V_{\mathbf{B}^\perp}^{\tau^{\mathrm{c}}} |u_{\tau^{\mathrm{c}}}|^2 \,\mathrm{d}x_1\,\mathrm{d}x_2 = 0\,.$$

When $\mathbf{B}^\perp \neq 0$ (that means when the magnetic field is not tangent to the axis of the wedge), we have $V_{\mathbf{B}^\perp}^{\tau^{\mathrm{c}}} > 0$ a.e. and we deduce $u_{\tau^{\mathrm{c}}} = 0$, that is a contradiction.

Assume now that $\mathbf{B}^\perp = 0$, then $V_{\mathbf{B}^\perp}^{\tau}(x_1, x_2) = \tau^2$ and therefore $\tau^{\mathrm{c}} = 0$. Denote by $\rho_{\tau^{\mathrm{c}}} := |u_{\tau^{\mathrm{c}}}|$, due to the standard diamagnetic inequality (see [28]), it satisfies

$$\int_{\mathcal{S}_\alpha} |\nabla \rho_{\tau^{\mathrm{c}}}|^2 = 0\,.$$

and therefore $\rho_{\tau^{\mathrm{c}}} = 0$ a.e. that is a contradiction. □

Together with the continuity result Theorem 4.5 of the next section, this shows that the infimum of the local ground state energy of the semiclassical magnetic Laplacian along edges (see Section 1.1 and Section 1.3) does not vanish. Notice that there is no hope of proving a uniform lower bound for $E(\mathbf{B}, \mathcal{W}_\alpha)$ since it goes to 0 for a magnetic field tangent to a face when $\alpha \to 0$ ([42, Section 5]).

4. Regularity of the ground state energy

In this section we prove the continuity of the application $(\mathbf{B}, \alpha) \mapsto E(\mathbf{B}, \mathcal{W}_\alpha)$. The domain of the quadratic form $\mathcal{Q}^{\tau}_{\mathbf{B},\alpha}$ depends on the geometry (see (2.6)), moreover the bottom of the spectrum of the operator $H(\mathbf{A}^{\|}, \mathcal{S}_\alpha) + V^{\tau}_{\mathbf{B}^\perp}$ may be essential, see Subsection 2.3. Therefore we cannot apply directly Kato's perturbation theory.

In this section we use the generic notation g (like geometry) for a couple $(\mathbf{B}, \alpha) \in \mathbb{S}^2 \times (0, 2\pi)$. We denote by $E(g) := E(\mathbf{B}, \mathcal{W}_\alpha)$ and $s(g; \tau) := s(\mathbf{B}, \mathcal{S}_\alpha; \tau)$. We also note $\mathcal{Q}^{\tau}_{g}$ the quadratic form $\mathcal{Q}^{\tau}_{\mathbf{B},\alpha}$

4.1. **Uniform Agmon estimates.** Here we give Agmon's estimates of concentration for the eigenfunctions of the operator $H(\mathbf{A}^{\|},\mathcal{S}_\alpha)+V^{\tau}_{\mathbf{B}^\perp}$ associated with the ground state energy $E(g)$.

First we recall a basic commutator formula (see [16, Chapter 3]):

Lemma 4.1. *Let Φ be a uniformly Lipschitz function on $\mathcal{S}_\alpha$ and let (E,u) be an eigenpair of the operator $H(\mathbf{A}^{\|},\mathcal{S}_\alpha)+V^{\tau}_{\mathbf{B}^\perp}$. Then we have*

$$\forall u\in \mathrm{Dom}(\mathcal{Q}^\tau_g),\quad \mathcal{Q}^\tau_g(e^\Phi u)=\int_{\mathcal{S}_\alpha}\left(E+|\nabla\Phi|^2\right)e^{2\Phi}|u|^2\,. \tag{4.1}$$

We introduce the lowest energy of $H(\mathbf{A}^{\|},\mathcal{S}_\alpha)+V^{\tau}_{\mathbf{B}^\perp}$ far from the origin:

$$\widetilde{\mathscr{E}}^*(g):=\begin{cases}\mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha) & \text{if}\quad \alpha\neq\pi\,,\\ E(\mathbf{B},\mathcal{W}_\alpha) & \text{if}\quad \alpha=\pi\,.\end{cases} \tag{4.2}$$

We have $\widetilde{\mathscr{E}}^*(g)=\sigma(\theta^0)$ where θ^0 is the minimum angle between the magnetic field and the boundary of $\mathcal{W}_\alpha$. Since $\theta\mapsto\sigma(\theta)$ is Lipschitz continuous we deduce that $g\mapsto\widetilde{\mathscr{E}}^*(g)$ is Lipschitz continuous on $\mathbb{S}^2\times(0,2\pi)$.

Denote by

$$\delta(g):=\widetilde{\mathscr{E}}^*(g)-E(g) \tag{4.3}$$

and recall that when $\delta(g)>0$ we can apply Theorem 3.6. The following proposition gives the exponential decay for the first eigenfunctions of $H(\mathbf{A}^{\|},\mathcal{S}_\alpha)+V^{\tau(g)}_{\mathbf{B}^\perp}$, provided that $\delta(g)>0$, including the precise control of the decay depending on $\delta(g)$:

Proposition 4.2. *Let $g=(\mathbf{B},\alpha)\in\mathbb{S}^2\times(0,2\pi)$ and $\delta(g)$ defined in (4.3). We suppose that $\delta(g)>0$. Let $\tau(g)\in\mathbb{R}$ be a value of the Fourier parameter given in Theorem 3.6 such that $s(g;\tau(g))=E(g)$. For $\nu\in(0,\sqrt{\delta(g)})$ let $\phi_\nu(x_1,x_2):=\nu\sqrt{x_1^2+x_2^2}$ be an Agmon distance. Then there exist universal constants $C>0$ and $C_1>0$ such that for all eigenfunctions u_g of $H(\mathbf{A}^{\|},\mathcal{S}_\alpha)+V^{\tau(g)}_{\mathbf{B}^\perp}$ associated with $E(g)$ we have*

$$\mathcal{Q}^{\tau(g)}_g(e^{\phi_\nu}u_g)\leqslant C\frac{1}{\delta(g)-\nu^2}e^{f(\delta(g),\nu,\alpha)}\|u_g\|^2_{L^2} \tag{4.4}$$

where

$$f(\delta,\nu,\alpha)=C_1\frac{\nu}{\alpha\sqrt{\delta-\nu^2}}\,. \tag{4.5}$$

Proof. We know from the results of [1] that $e^{\phi_\nu}u_g\in L^2(\mathcal{S}_\alpha)$. Since $|\nabla\phi_\nu|^2=\nu^2$ the commutator formula (4.1) provides

$$\int_{\mathcal{S}_\alpha}(E(g)+\nu^2)e^{2\phi_\nu}|u_g|^2=\mathcal{Q}^{\tau(g)}_g(e^{\phi_\nu}u_g)\,. \tag{4.6}$$

We use cut-off functions $\chi_{1,R}$ and $\chi_{2,R}$ in $C^\infty(\mathcal{S}_\alpha,[0,1])$ that satisfy $\chi_{1,R}(x)=0$ when $|x|\geqslant 2R$ and $\chi_{1,R}(x)=1$ when $|x|\leqslant R$ and $\chi_{1,R}^2+\chi_{2,R}^2=1$. We also assume without restriction that there exists $C > 0$ such that

$$\forall R > 0, \quad \sum_{j=1}^{2} |\nabla \chi_{j,R}|^2 \leqslant \frac{C}{R^2} \,. \tag{4.7}$$

Lemma 3.2 provides

$$\mathcal{Q}_g^{\tau(g)}(e^{\phi_\nu} u_g) = \sum_{j=1}^{2} \mathcal{Q}_g^{\tau(g)}(\chi_{j,R} e^{\phi_\nu} u_g) - \sum_{j=1}^{2} \|\nabla \chi_{j,R} e^{\phi_\nu} u_g\|^2$$

and from (4.6) and (4.7) we get

$$\left(E(g) + \nu^2 + \frac{C}{R^2}\right) \|e^{\phi_\nu} u_g\|_{L^2}^2 \geqslant \sum_{j=1}^{2} \mathcal{Q}_g^{\tau(g)}(\chi_{j,R} e^{\phi_\nu} u_g) \,.$$

Note that since $\delta(g) > 0$ we have $\alpha \neq \pi$ and Lemma 3.3 provides a universal constant $C_1 > 0$ such that $\mathcal{Q}_g^{\tau(g)}(\chi_{2,R} u_g) \geqslant (\widetilde{\mathscr{E}}^*(g) - \frac{C_1}{\alpha^2 R^2}) \|\chi_{2,R} e^{\phi_\nu} u_g\|_{L^2}^2$. We deduce another constant $C > 0$ such that

$$\begin{aligned}\left(\delta(g) - \nu^2 - \frac{C}{R^2}(1 + \tfrac{1}{\alpha^2})\right) \|\chi_{2,R} e^{\phi_\nu} u_g\|_{L^2}^2 \\ \leqslant \left(E(g) + \nu^2 + \frac{C}{R^2}(1 + \tfrac{1}{\alpha^2})\right) \|\chi_{1,R} e^{\phi_\nu} u_g\|_{L^2}^2 \,.\end{aligned} \tag{4.8}$$

We now choose $R > 0$ such that:

$$\frac{C}{R^2}(1 + \tfrac{1}{\alpha^2}) = \frac{\delta(g) - \nu^2}{2} \tag{4.9}$$

Moreover since $E(g) < \widetilde{\mathscr{E}}^*(g) \leqslant 1$ and $\nu \in (0, \sqrt{\delta(g)})$ we have $E(g) + \nu^2 + \frac{C}{R^2}(1 + \frac{1}{\alpha^2}) \leqslant 2$. We deduce from (4.8)

$$\|e^{\phi_\nu} u_g\|_{L^2}^2 \leqslant \left(\frac{4}{\delta(g) - \nu^2} + 1\right) \|\chi_{1,R} e^{\phi_\nu} u_g\|_{L^2}^2 \leqslant \left(\frac{4}{\delta(g) - \nu^2} + 1\right) e^{4\nu R} \|u_g\|_{L^2}^2 \,. \tag{4.10}$$

Notice now that due to the choice of R in (4.9) we have $\nu R \leqslant C_1 \frac{1}{\alpha} \frac{\nu}{\sqrt{\delta(g) - \nu^2}} := f(\delta(g), \nu, \alpha)$ with $C_1 > 0$. Recall that $\delta(g) \leqslant 1$, we get another constant $C > 0$ such that

$$\|e^{\phi_\nu} u_g\|_{L^2}^2 \leqslant C \left(\frac{1}{\delta(g) - \nu^2}\right) e^{f(\delta(g), \nu, \alpha)} \|u_g\|_{L^2}^2 \,. \tag{4.11}$$

We deduce the estimates on the quadratic form from the identity (4.6). □

This Agmon estimates will be applied to different situations: either a set of $g = (\mathbf{B}, \alpha)$ such that $\delta(g)$ is uniformly bounded from below (and therefore for a fix $\nu \in (0, \sqrt{\delta(g)})$, the above estimate will be uniform with respect to g); either a set of g such that $\delta(g)$ tends to 0, and in that case we will choose $\nu = \frac{1}{2}\sqrt{\delta(g)}$ and use the explicit control with respect to $\delta(g)$. In both cases we will ask the opening angle α to be not too small.

4.2. **Polar coordinates.** Let $(r,\psi)\in\mathbb{R}_+\times(-\frac{\alpha}{2},\frac{\alpha}{2})$ be the usual polar coordinates of $\mathcal{S}_\alpha$. We use the change of variables associated with the normalized polar coordinates $(r,\phi):=(r,\frac{\psi}{\alpha})\in\Omega_0:=\mathbb{R}_+\times(-\frac{1}{2},\frac{1}{2})$. After a change a gauge (see [8, Section 3] and [42, Section 5]) we get that the quadratic form $\mathcal{Q}_g^\tau$ is unitary equivalent to the quadratic form

$$\widetilde{\mathcal{Q}}_g^\tau(u):=\int_{\Omega_0}\left(|(\partial_r-i\alpha r\phi b_3)u|^2+\frac{1}{\alpha^2r^2}|\partial_\phi u|^2+\widetilde{V}_g^\tau(r,\phi)|u|^2\right)r\,\mathrm{d}r\,\mathrm{d}\phi \tag{4.12}$$

with the electric potential in polar coordinates:

$$\widetilde{V}_g^\tau(r,\phi):=\big(r\cos(\phi\alpha)b_2-r\sin(\phi\alpha)b_1-\tau\big)^2\,. \tag{4.13}$$

The form domain is

$$\mathrm{Dom}(\widetilde{\mathcal{Q}}_g^\tau)=\left\{u\in L^2_r(\Omega_0),\ (\partial_r-i\alpha r\phi b_3)u\in L^2_r(\Omega_0),\ \frac{1}{r}\partial_\phi u\in L^2_r(\Omega_0),\ \sqrt{\widetilde{V}_g^\tau}\,u\in L^2_r(\Omega_0)\right\}$$

where $L^2_r(\Omega_0)$ stands for the set of the square-integrable functions for the weight $r\,\mathrm{d}r$.

Notation 4.3. Let $g_0=(\mathbf{B}_0,\alpha_0)\in\mathbb{S}^2\times(0,2\pi)$ and $\eta>0$. We denote by $B(g_0,\eta)$ the ball of $\mathbb{S}^2\times\mathbb{R}$ of center g_0 and of radius η related to the euclidean norm $\|g\|:=(\|\mathbf{B}\|_2^2+\alpha^2)^{1/2}$.

Lemma 4.4. *Let $g_0=(\mathbf{B}_0,\alpha_0)\in\mathbb{S}^2\times(0,2\pi)$. There exist $C>0$ and $\eta>0$ such that $B(g_0,\eta)\subset\mathbb{S}^2\times(0,2\pi)$ and for all $g\in B(g_0,\eta)$ we have for all $u\in\mathrm{Dom}(\widetilde{\mathcal{Q}}_g^\tau)\cap\mathrm{Dom}(\widetilde{\mathcal{Q}}_{g_0}^\tau)$:*

$$\forall\tau\in\mathbb{R},\quad \widetilde{\mathcal{Q}}_g^\tau(u)\leqslant\widetilde{\mathcal{Q}}_{g_0}^\tau(u)+C\|g-g_0\|\left(\|ru\|^2_{L^2_r(\Omega_0)}+\widetilde{\mathcal{Q}}_{g_0}^\tau(u)\right)\,.$$

Proof. Let $g_0=(\mathbf{B}_0,\alpha_0)$ and $g=(\mathbf{B},\alpha)$ be in $\mathbb{S}^2\times(0,2\pi)$. We denote by $(b_{j,0})_j$ and $(b_j)_j$ the cartesian coordinates of $\mathbf{B}_0$ and $\mathbf{B}$. Let $d:=\|g-g_0\|$. We discuss separately the three terms of $\widetilde{\mathcal{Q}}_g^\tau(u)$ written in (4.12). For the first one we write

$$\begin{aligned}|(\partial_r-i\alpha b_3r\phi)u|^2\leqslant&|(\partial_r-i\alpha_0b_{3,0}r\phi)u|^2\\&+|\alpha_0b_{3,0}-\alpha b_3|^2r^2|u|^2+2r|u||\alpha_0b_{3,0}-\alpha b_3||(\partial_r-i\alpha_0b_{3,0}r\phi)u|\end{aligned}$$

We have $|\alpha_0b_{3,0}-\alpha b_3|\leqslant\alpha_0|b_{3,0}-b_3|+|b_3||\alpha_0-\alpha|\leqslant(2\pi+1)d$, and

$$2r|u||\alpha_0b_{3,0}-\alpha b_3||(\partial_r-i\alpha_0b_{3,0}r\phi)u|\leqslant(2\pi+1)d\left(r^2|u|^2+|(\partial_r-i\alpha_0b_{3,0}r\phi)u|^2\right)\,.$$

Therefore there exists $C_1>0$ such that for all $g\in\mathbb{S}^2\times(0,2\pi)$:

$$\begin{aligned}|(\partial_r-i\alpha b_3r\phi)u|^2\leqslant&|(\partial_r-i\alpha_0b_{3,0}r\phi)u|^2\\&+C_1d\left(r^2|u|^2(1+d)+|(\partial_r-i\alpha_0b_{3,0}r\phi)u|^2\right)\,.\end{aligned} \tag{4.14}$$

We deal with the second term: we have

$$\left|\frac{1}{\alpha^2r^2}|\partial_\phi u|^2-\frac{1}{\alpha_0^2r^2}|\partial_\phi u|^2\right|=d\frac{\alpha+\alpha_0}{\alpha^2\alpha_0}\frac{1}{\alpha_0r^2}|\partial_\phi u|^2\,.$$

Therefore there exist $\eta>0$ and $C_2>0$ such that $B(g_0,\eta)\subset\mathbb{S}^2\times(0,2\pi)$ and

$$\forall g\in B(g_0,\eta),\quad\frac{1}{\alpha^2r^2}|\partial_\phi u|^2\leqslant\frac{1}{\alpha_0^2r^2}|\partial_\phi u|^2+C_2d\frac{1}{\alpha_0r^2}|\partial_\phi u|^2\,. \tag{4.15}$$

For the third term we write

$$\begin{aligned}\widetilde{V}_{g}^{\tau}(r,\phi) \leqslant & \widetilde{V}_{g_0}^{\tau}(r,\phi) + |\cos(\alpha\phi)b_2 - \cos(\alpha_0\phi)b_{2,0} + \sin(\alpha_0\phi)b_{1,0} - \sin(\alpha\phi)b_1|^2 r^2 \\ & + 2\sqrt{\widetilde{V}_{g_0}^{\tau}(r,\phi)}\,|\cos(\alpha\phi)b_2 - \cos(\alpha_0\phi)b_{2,0} + \sin(\alpha_0\phi)b_{1,0} - \sin(\alpha\phi)b_1|\,r\,.\end{aligned}$$

We get $C_3 > 0$ and $C_4 > 0$ such that for all $g \in \mathbb{S}^2 \times (0, 2\pi)$ and for all $\tau \in \mathbb{R}$:

$$\forall (r,\phi) \in \Omega_0, \quad \widetilde{V}_{g,\tau}(r,\phi) \leqslant (1 + C_3 d)\widetilde{V}_{g_0}^{\tau}(r,\phi) + C_4 r^2 d\,. \tag{4.16}$$

Combining (4.14), (4.15) and (4.16) we get $C > 0$ such that for all $g \in B(g_0, \eta)$:

$$\widetilde{\mathcal{Q}}_g^{\tau}(u) \leqslant \widetilde{\mathcal{Q}}_{g_0}^{\tau}(u)| + C\|g - g_0\| \left(\|ru\|^2_{L^2_r(\Omega_0)} + \widetilde{\mathcal{Q}}_{g_0}^{\tau}(u)\right)\,.$$

□

4.3. **Continuity.**

Theorem 4.5. *The function $g \mapsto E(g)$ is continuous on $\mathbb{S}^2 \times (0, 2\pi)$.*

Proof. Let $g_0 \in \mathbb{S}^2 \times (0, 2\pi)$. We distinguish different cases depending on whether (3.9) is strict or not. Recall that $\delta(g)$ is defined in (4.3).

Case 1: When

$$\delta(g_0) > 0\,. \tag{4.17}$$

Let us note that in that case $\alpha_0 \neq \pi$ (see (4.2)). We use Theorem 3.6: There exists $\tau^{\mathrm{c}} \in \mathbb{R}$ such that the band function $\tau \mapsto s(g_0;\tau)$ reaches its infimum in τ^{c} and there exists a normalized eigenfunction (in polar coordinate) u_0 for $\widetilde{\mathcal{Q}}_{g_0}^{\tau^{\mathrm{c}}}$ with exponential decay in r. We use this function as a quasimode for $\widetilde{\mathcal{Q}}_{g}^{\tau^{\mathrm{c}}}$. We get from Lemma 4.4 constants $C > 0$ and $\eta > 0$ such that for all $g \in B(g_0, \eta)$:

$$\begin{aligned}\widetilde{\mathcal{Q}}_{g}^{\tau^{\mathrm{c}}}(u_0) &\leqslant \widetilde{\mathcal{Q}}_{g_0}^{\tau^{\mathrm{c}}}(u_0) + C\|g - g_0\| \left(\|ru_0\|^2_{L^2_r} + \widetilde{\mathcal{Q}}_{g_0}^{\tau^{\mathrm{c}}}(u_0)\right) \\ &= E(g_0) + C\|g - g_0\| \left(\|ru_0\|^2_{L^2_r} + E(g_0)\right)\end{aligned}$$

and therefore the min-max principle and relation (2.3) provide

$$E(g) \leqslant E(g_0) + C\|g - g_0\| \left(\|ru_0\|^2_{L^2_r} + E(g_0)\right)\,. \tag{4.18}$$

Since u_0 has exponential decay in r we get

$$\limsup_{g \to g_0} E(g) \leqslant E(g_0)\,. \tag{4.19}$$

Using this upper bound, the assumption (4.17) and the continuity of $g \mapsto \widetilde{\mathscr{E}}^*(g)$, we deduce that there exist $\kappa > 0$ and $\epsilon_0 > 0$ such that $\overline{B(g_0,\kappa)} \subset \mathbb{S}^2 \times ((0, 2\pi)\backslash\pi)$ and

$$\forall g \in B(g_0,\kappa), \quad \epsilon_0 < \delta(g)\,. \tag{4.20}$$

Let $g \in B(g_0, \kappa)$, Theorem 3.6 provides $\tau(g) \in \mathbb{R}$ such that $s(g;\tau(g)) = E(g)$ is a discrete eigenvalue for the operator $H(\mathbf{A}^{\|}, \mathcal{S}_\alpha) + V_{\mathbf{B}^\perp}^{\tau(g)}$. We denote by u_g an associated normalized eigenfunction in polar coordinates. We use (4.20) and Proposition 4.2: For a fix $\nu \in (0, \sqrt{\epsilon_0})$ there exists $C_0 > 0$ such that

$$\forall g \in B(g_0, \kappa), \quad \|e^{\nu\, r} u_g\|_{L^2_r(\Omega_0)} < C_0 \,. \tag{4.21}$$

We use u_g as a quasimode for $\widetilde{\mathcal{Q}}_{g_0}^{\tau(g)}$: (4.21) and Lemma 4.4 yields

$$\exists C_1 > 0, \ \forall g \in B(g_0, \kappa), \quad \widetilde{\mathcal{Q}}_{g_0}^{\tau(g)}(u_g) \leqslant \widetilde{\mathcal{Q}}_{g}^{\tau(g)}(u_g) + C_1 \|g - g_0\| \tag{4.22}$$

and since u_g satisfies $\widetilde{\mathcal{Q}}_{g}^{\tau(g)}(u_g) = E(g)$ we deduce from the min-max principle and (2.3):

$$\forall g \in B(g_0, \kappa), \quad E(g_0) \leqslant E(g) + C_1 \|g - g_0\| \,. \tag{4.23}$$

This last upper bound combined with (4.19) brings the continuity of $E(\cdot)$ in g_0 when $E(g_0) < \widetilde{\mathscr{E}}^*(g_0)$.

Case 2: When

$$\delta(g_0) = 0 \,. \tag{4.24}$$

Let us suppose that for all $\epsilon > 0$ there exists $\kappa > 0$ such that for all $g \in B(g_0, \kappa)$ we have

$$\widetilde{\mathscr{E}}^*(g) - \epsilon \leqslant E(g) \leqslant \widetilde{\mathscr{E}}^*(g) \,.$$

In that case we deduce the continuity of $E(\cdot)$ in g_0 from the continuity of $\widetilde{\mathscr{E}}^*(\cdot)$.

Let us write the contraposition of the previous statement and exhibit a contradiction. We suppose that there exists $\epsilon_0 > 0$ such that for all $\kappa > 0$ there exists $g \in \mathbb{S}^2 \times (0, 2\pi)$ satisfying $\|g - g_0\| < \kappa$ and $E(g) < \widetilde{\mathscr{E}}^*(g) - \epsilon_0$. This implies $\alpha \neq \pi$ (see (4.2)). Theorem 3.6 provides $\tau(g) \in \mathbb{R}$ such that $E(g) = s(g; \tau(g))$ and we denote by u_g an associated normalized eigenfunction for $H(\mathbf{A}^{\parallel}, \mathcal{S}_\alpha) + V_{\mathbf{B}^\perp}^{\tau(g)}$. Again Proposition 4.2 shows that this eigenfunction has exponential decay uniformly in g: For each $\nu \in (0, \sqrt{\epsilon_0})$, we have $C_0 > 0$ that does not depend on g such that

$$\|e^{\nu\, r} u_g\|_{L^2_r(\Omega_0)} < C_0 \,.$$

We use u_g as a quasimode for $\widetilde{\mathcal{Q}}_{g_0}^{\tau(g)}$: There exists a constant $C_1 > 0$ that does not depend on g such that

$$\begin{aligned} \widetilde{\mathcal{Q}}_{g_0}^{\tau(g)}(u_g) &\leqslant \widetilde{\mathcal{Q}}_{g}^{\tau(g)}(u_g) + C_1 \|g - g_0\| \\ &< \widetilde{\mathscr{E}}^*(g) - \epsilon_0 + C_1 \kappa. \end{aligned}$$

The min-max Principle and (2.3) provide

$$E(g_0) < \widetilde{\mathscr{E}}^*(g) - \epsilon_0 + C_1 \kappa \,.$$

Let $\epsilon > 0$, the continuity of $\widetilde{\mathscr{E}}^*(\cdot)$ implies that for $\kappa > 0$ small enough there holds $\widetilde{\mathscr{E}}^*(g) < \widetilde{\mathscr{E}}^*(g_0) + \epsilon$. We have proved:

$$\exists \epsilon_0 > 0, \, \exists C_1 > 0, \, \forall \epsilon > 0, \, \exists \kappa_0 > 0, \forall \kappa \in (0, \kappa_0), \quad E(g_0) < \widetilde{\mathscr{E}}^*(g_0) - \epsilon_0 + C_1 \kappa + \epsilon \,.$$

Choosing $\epsilon > 0$ and $\kappa > 0$ small enough we get a contradiction with (4.24). $\square$

4.4. **Lipschitz and Hölder continuity.** Looking at the previous proof, and more particularly (4.18) and (4.23), we get:

Proposition 4.6. *Assume that* $g_0 \in \mathbb{S}^2 \times (0, 2\pi)$ *is such that* $E(g_0) < \widetilde{\mathscr{E}}^*(g_0)$*. Then* $g \mapsto E(g)$ *is Lipschitz in a neighborhood of* g_0*.*

Remark that the hypothesis of the above proposition applies only for $\alpha \neq \pi$.

When $E(g_0) = \widetilde{\mathscr{E}}^*(g_0)$, the situation is more complicated, indeed for g close to g_0 such that $E(g) < \widetilde{\mathscr{E}}^*(g)$ we do not have a uniform lower bound on $\delta(g)$ and the exponential decay of the eigenfunction u_g (used as a quasi-mode in (4.22)) becomes worse and worse, see Proposition 4.2. Therefore we do not know any uniform upper bound for the error term $\|ru_g\|$. To solve the situation we follow the dependence upon $\delta(g)$ of the constants appearing in the estimation of $\|ru_g\|$.

Proposition 4.7. *Assume that* $g_0 \in \mathbb{S}^2 \times (0, 2\pi)$ *is such that* $E(g_0) = \widetilde{\mathscr{E}}^*(g_0)$*. Then* $g \mapsto E(g)$ *is* $\frac{1}{3}$*-Hölder in a neighborhood of* g_0*.*

Proof. Since $\theta \mapsto \sigma(\theta)$ is Lipschitz on $[0, 2\pi)$, using (1.6) we restrict to the $g \neq g_0$ that lie in a small neighborhood of g_0 such that $E(g) < \widetilde{\mathscr{E}}^*(g)$. Denote by $V^*(g_0)$ such a set. Assume that $g \in V^*(g_0)$ satisfies $E(g) \geqslant E(g_0)$ then there holds

$$0 \leqslant E(g) - E(g_0) \leqslant \widetilde{\mathscr{E}}^*(g) - \widetilde{\mathscr{E}}^*(g_0)$$

and one gets $0 \leqslant E(g) - E(g_0) \leqslant C\|g - g_0\|$ with $C > 0$ that does not depend on g since we know that $g \mapsto \widetilde{\mathscr{E}}^*(g)$ is Lipschitz. We now have to deal with the case $g \in V^*(g_0)$ and $E(g) \leqslant E(g_0)$. Denote by $\tau(g) \in \mathbb{R}$ a minimizer for the band function $\tau \mapsto s(g, \tau)$ and u_g an associated normalized eigenfunction as in the proof of Theorem 4.5. Noticing that

$$\forall \nu > 0, \forall r > 0, \quad (r\nu)^2 e^{-2\nu r} \leqslant 1$$

we get

$$\forall \nu \in (0, \sqrt{\delta(g)}\,), \quad \|ru_g\|^2 \leqslant \nu^{-2}\|e^{\nu r}u_g\|^2 . \tag{4.25}$$

We set $\nu := \frac{1}{2}\sqrt{\delta(g)}$ and we get $f(\delta(g), \nu, \alpha) = \frac{C_1}{\sqrt{3}}$ (see (4.5)). We deduce from (4.11) and (4.25) a constant $C > 0$ such that

$$\forall g \in V^*(g_0), \quad \|ru_g\|^2 \leqslant C(\delta(g))^{-2} . \tag{4.26}$$

Combining Lemma 4.4 with (4.26) we get

$$\widetilde{\mathcal{Q}}_{g_0}^{\tau(g)}(u_g) \leqslant E(g) + C\|g - g_0\| \left(C(\delta(g))^{-2} + E(g)\right)$$

and using the min-max principle we get $C_0 > 0$ such that

$$0 \leqslant (E(g_0) - E(g))\delta(g)^2 \leqslant C_0\|g - g_0\| .$$

Writing $\delta(g) = E(g_0) - E(g) - (\widetilde{\mathscr{E}}^*(g_0) - \widetilde{\mathscr{E}}^*(g))$ and using that $\widetilde{\mathscr{E}}^*$ is Lipschitz, we get another constant $C_2 > 0$ such that

$$0 \leqslant (E(g_0) - E(g))^3 \leqslant C_2\|g - g_0\|$$

and therefore $E(g)$ is $\frac{1}{3}$-Hölder in a neighborhood of g_0.

□

Remark 4.8. These regularity results are obtained for unitary constant magnetic fields. Using the scaling (1.1), these results are easily extended to any non-zero constant magnetic fields.

5. UPPER BOUND FOR SMALL ANGLES

5.1. **An auxiliary problem on a half-line.** Let $L^2_r(\mathbb{R}_+)$ be the space of the square-integrable functions for the weight $r\,\mathrm{d}r$ and let

$$B^1_r(\mathbb{R}_+) := \{u \in L^2_r(\mathbb{R}^+), u' \in L^2_r(\mathbb{R}^+), ru \in L^2_r(\mathbb{R}^+)\}\ .$$

We define the 1d quadratic form

$$\mathfrak{q}_\tau(u) := \int_{\mathbb{R}_+} \left(|u'(r)|^2 + (r-\tau)^2|u(r)|^2\right) r\,\mathrm{d}r$$

on the domain $B^1_r(\mathbb{R}_+)$. As we will see later, if u is a function of $L^2(\mathcal{S}_\alpha)$ that does not depend on the angular variable and if $b_2 \neq 0$, $b_2^{-1}\mathcal{Q}^\tau_{\mathbf{B},\alpha}(u)$ written in polar coordinates degenerates formally toward $\mathfrak{q}_\tau(u)$ when α goes to 0.

We denote by $\mathfrak{g}_\tau$ the Friedrichs extension of the quadratic form $\mathfrak{q}_\tau$. This operator has been introduced in [50] and studied in [41] as the reduced operator of a 3d magnetic Hamiltonian with axisymmetric potential.

The technics from [7] show that $\mathfrak{g}_\tau$ has compact resolvent. We denote by $\zeta(\tau)$ its first eigenvalue. For all $\tau \in \mathbb{R}$, $\zeta(\tau)$ is a simple eigenvalue and we denote by z_τ an associated normalized eigenfunction. Basic estimates of Agmon show that z_τ has exponential decay. The following properties are shown in [41]:

The function $\tau \mapsto \zeta(\tau)$ reaches its infimum. We denote by

$$\Xi_0 := \inf_{\tau\in\mathbb{R}} \zeta(\tau) \tag{5.1}$$

the infimum. Let $\tau_0 > 0$ be the lowest real number such that $\zeta(\tau_0) = \Xi_0$. We have

$$\Theta_0 < \Xi_0 \leqslant \sqrt{4-\pi}\ . \tag{5.2}$$

Numerical simulations show that $\Xi_0 \approx 0.8630$.

5.2. **Upper bounds and consequences.** Let $\mathbf{B} = (b_1, b_2, b_3)$ be a magnetic field in $\mathbb{S}^2$. Due to symmetry we assume $b_j \geqslant 0$ for all $j \in \{1,2,3\}$ (see Proposition 2.1). Recall the quadratic form $\widetilde{\mathcal{Q}}^\tau_{\mathbf{B},\alpha}$ associated with $H(\mathbf{A}^\parallel, \mathcal{S}_\alpha) + V^\tau_{\mathbf{B}^\perp}$ in polar coordinates (see (4.12)). The injection from $\left(B^1_r(\mathbb{R}_+), \|\cdot\|_{L^2_r(\mathbb{R}_+)}\right)$ into $\left(\mathrm{Dom}(\widetilde{\mathcal{Q}}^\tau_{\mathbf{B},\alpha}(u)), \|\cdot\|_{L^2_r(\Omega_0)}\right)$ is an isometry, therefore we can restrict $\widetilde{\mathcal{Q}}^\tau_{\mathbf{B},\alpha}$ to $B^1_r(\mathbb{R}_+)$ and in the following for $u \in B^1_r(\mathbb{R}_+)$ we denote again by u the associated function defined on Ω_0. Assume $b_2 > 0$, that means that the magnetic field is not tangent to the symmetry plane of the wedge. For $u \in B^1_r(\mathbb{R}_+)$ we have formally that $b_2^{-1}\widetilde{\mathcal{Q}}^{\tau\sqrt{b_2}}_{\mathbf{B},\alpha}(u)$ goes to $\mathfrak{q}_\tau(u)$ when α goes to 0.

The following lemma makes this argument more rigorous:

Lemma 5.1. *Let* $\mathbf{B} \in \mathbb{S}^2$ *with* $b_2 > 0$. *For* $u \in B^1_r(\mathbb{R}_+)$ *we denote by* $u^{\mathrm{sc}}(r) := b_2^{1/2} u(b_2^{1/2} r)$ *the associated rescaled function. We have* $\|u^{\mathrm{sc}}\|_{L^2_r(\mathbb{R}_+)} = \|u\|_{L^2_r(\mathbb{R}_+)}$ *and*

$$\widetilde{\mathcal{Q}}^{\tau\sqrt{b_2}}_{\mathbf{B},\alpha}(u^{\mathrm{sc}}) = b_2 \mathfrak{q}_\tau(u) + \frac{\alpha^2}{12}\|ru\|^2_{L^2_r(\mathbb{R}_+)}\frac{b_3^2}{b_2} + \frac{1}{2}(1-\operatorname{sinc}\alpha)\|ru\|^2_{L^2_r(\mathbb{R}_+)}\frac{b_1^2-b_2^2}{b_2} + 2\tau b_2\left(1-\operatorname{sinc}\frac{\alpha}{2}\right)\|\sqrt{r}u\|^2_{L^2_r(\mathbb{R}_+)} \, . \tag{5.3}$$

with $\operatorname{sinc}\alpha := \frac{\sin\alpha}{\alpha}$.

Proof. We evaluate $\widetilde{\mathcal{Q}}^{\tau}_{\mathbf{B},\alpha}(u)$ for $u \in B^1_r(\mathbb{R}_+)$:

$$\widetilde{\mathcal{Q}}^{\tau}_{\mathbf{B},\alpha}(u) = \int_{\mathbb{R}_+}\left(|u'(r)|^2 + (rb_2-\tau)|u(r)|^2\right) r\,\mathrm{d}r + \int_{\Omega_0}\alpha^2\phi^2 b_3^2 r^2 |u(r)|^2 r\,\mathrm{d}r\,\mathrm{d}\phi + \int_{\Omega_0}\left(\widetilde{V}^{\tau}_{\mathbf{B}^\perp}(r,\phi) - (rb_2-\tau)^2\right)|u(r)|^2 r\,\mathrm{d}r\,\mathrm{d}\phi \, .$$

We have

$$\int_{\Omega_0}\alpha^2 r^2\phi^2 b_3^2 |u(r)|^2 r\,\mathrm{d}r\,\mathrm{d}\phi = \frac{\alpha^2}{12}\|ru\|^2_{L^2_r(\mathbb{R}_+)} b_3^2 \, . \tag{5.4}$$

Elementary computations yield:

$$\widetilde{V}^{\tau}_{\mathbf{B}^\perp}(r,\phi) - (rb_2-\tau)^2 = r^2\sin^2(\alpha\phi)(b_1^2-b_2^2) - 2rb_2\tau\left(\cos(\alpha\phi)-1\right)^2 - 2rb_1\sin(\alpha\phi)\left(rb_2\cos(\alpha\phi)-\tau\right) \, .$$

Since the term $-2rb_1\sin(\alpha\phi)\left(rb_2\cos(\alpha\phi)-\tau\right)$ is odd with respect to ϕ, its integral on Ω_0 vanishes. For the other terms we use:

$$\int_{-1/2}^{1/2}\sin^2(\alpha\phi)\,\mathrm{d}\phi = \tfrac{1}{2}(1-\operatorname{sinc}\alpha) \quad \text{and} \quad \int_{-1/2}^{1/2}\big(\cos(\alpha\phi)-1\big)\,\mathrm{d}\phi = \operatorname{sinc}\tfrac{\alpha}{2} - 1 \, .$$

We deduce for all $u \in B^1_r(\mathbb{R}_+)$ and $\tau \in \mathbb{R}$:

$$\int_{\Omega_0}\left(\widetilde{V}^{\tau}_{\mathbf{B}^\perp}(r,\phi) - (rb_2-\tau)^2\right)|u(r)|^2 r\,\mathrm{d}r\,\mathrm{d}\phi = \tfrac{1}{2}(1-\operatorname{sinc}\alpha)\|ru\|^2_{L^2_r(\mathbb{R}_+)}(b_1^2-b_2^2) + 2\tau\left(1-\operatorname{sinc}\tfrac{\alpha}{2}\right)\|\sqrt{r}u\|^2_{L^2_r(\mathbb{R}_+)} b_2$$

and therefore (note that we have make the change $\tau \to \tau\sqrt{b_2}$):

$$\widetilde{\mathcal{Q}}^{\tau\sqrt{b_2}}_{\mathbf{B},\alpha}(u) = \int_{\mathbb{R}_+}\left(|u'(r)|^2 + (rb_2-\tau\sqrt{b_2})^2|u(r)|^2\right) r\,\mathrm{d}r + \tfrac{\alpha^2}{12}\|ru\|^2_{L^2_r(\mathbb{R}_+)} b_3^2 + \tfrac{1}{2}(1-\operatorname{sinc}\alpha)\|ru\|^2_{L^2_r(\mathbb{R}_+)}(b_1^2-b_2^2) + 2\tau\left(1-\operatorname{sinc}\tfrac{\alpha}{2}\right)\|\sqrt{r}u\|^2_{L^2_r(\mathbb{R}_+)} b_2^{3/2} \, . \tag{5.5}$$

Let $u^{\mathrm{sc}}(r) := b_2^{1/2} u(b_2^{1/2} r)$. An elementary scaling provides

$$\int_{\mathbb{R}_+}\left(|(u^{\mathrm{sc}})'(r)|^2 + (rb_2-\tau\sqrt{b_2})^2|u^{\mathrm{sc}}(r)|^2\right) r\,\mathrm{d}r = b_2\mathfrak{q}_\tau(u) \, .$$

Moreover we have

$$\|ru^{\mathrm{sc}}\|^2_{L^2_r(\mathbb{R}_+)} = b_2^{-1}\|ru\|^2_{L^2_r(\mathbb{R}_+)} \quad \text{and} \quad \|\sqrt{r}u^{\mathrm{sc}}\|^2_{L^2_r(\mathbb{R}_+)} = b_2^{-1/2}\|\sqrt{r}u\|^2_{L^2_r(\mathbb{R}_+)}\,,$$

therefore we deduce (5.3) from (5.5). ☐

Proposition 5.2. *Let* $\mathbf{B} \in \mathbb{S}^2$ *with* $b_2 > 0$. *There exists* $C(\mathbf{B}) > 0$ *such that*

$$\forall \alpha \in (0,\pi), \quad E(\mathbf{B}, \mathcal{W}_\alpha) \leqslant b_2\Xi_0 + C(\mathbf{B})\alpha^2\,. \tag{5.6}$$

Proof. We recall that $z_\tau \in B^1_r(\mathbb{R}_+)$ is a normalized eigenfunction associated with $\zeta(\tau)$ for the operator $\mathfrak{g}_\tau$ (see Subsection 5.1). We define

$$z^{\mathrm{sc}}_{\tau_0}(r) := b_2^{1/2} z_{\tau_0}(rb_2^{1/2})$$

where $\tau_0 \in \mathbb{R}$ satisfies $\zeta(\tau_0) = \Xi_0$ (see (5.1)). For all $\alpha > 0$ we have:

$$0 \leqslant 1 - \operatorname{sinc}\alpha \leqslant \frac{\alpha^2}{6} \quad \text{and} \quad 0 \leqslant 1 - \operatorname{sinc}\frac{\alpha}{2} \leqslant \frac{\alpha^2}{24}\,. \tag{5.7}$$

We have $\mathfrak{q}_{\tau_0}(z_{\tau_0}) = \Xi_0$, therefore Lemma 5.1 and (5.7) provides

$$\widetilde{Q}^{\tau_0\sqrt{b_2}}_{\mathbf{B},\alpha}(z^{\mathrm{sc}}_{\tau_0}) \leqslant b_2\Xi_0 + \frac{\alpha^2}{12}\left(\frac{b_3^2 + |b_1^2 - b_2^2|}{b_2}\|rz_{\tau_0}\|^2_{L^2_r} + \tau_0 b_2\|\sqrt{r}z_{\tau_0}\|^2_{L^2_r(\mathbb{R}_+)}\right)\,.$$

Since $\|z^{\mathrm{sc}}_{\tau_0}\|_{L^2_r(\Omega_0)} = \|z_{\tau_0}\|_{L^2_r(\mathbb{R}_+)} = 1$ the min-max principle provides:

$$\exists C(\mathbf{B}) > 0,\ \forall \alpha \in (0,\pi), \quad s(\mathbf{B}, \mathcal{S}_\alpha; \tau_0\sqrt{b_2}) \leqslant b_2\Xi_0 + C(\mathbf{B})\alpha^2\,.$$

We deduce the proposition with (2.3). ☐

As a direct consequence we get

Corollary 5.3. *Let* $\mathbf{B} \in \mathbb{S}^2$ *with* $b_2 > 0$. *We have the following upper bound:*

$$\limsup_{\alpha\to 0} E(\mathbf{B}, \mathcal{W}_\alpha) \leqslant b_2\Xi_0\,. \tag{5.8}$$

Numerical computations show that $E(\mathbf{B}, \mathcal{W}_\alpha)$ seems to go to $b_2\Xi_0$ when α goes to 0 (see Section 6 and [40, Section 6.4]). This question remains open. However the upper bound (5.8) is sufficient to give a comparison between the spectral quantity associated with an edge and the one coming from regular model problem:

Theorem 5.4. *Let* $\mathbf{B} \in \mathbb{S}^2$ *with* $b_2 > 0$. *Then there exists* $\alpha(\mathbf{B}) \in (0,\pi)$ *such that for all* $\alpha \in (0, \alpha(\mathbf{B}))$ *we have* $E(\mathbf{B}, \mathcal{W}_\alpha) < \mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha)$.

Proof. We introduce $\theta^0 := \min\{\theta^+, \theta^-\}$ (θ^0 depends on α and $\mathbf{B}$). For $\alpha \in (0,\pi)$ we have $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) = \sigma(\theta^0)$. We recall the inequality from [23, Section 3.4]:

$$\sigma(\theta^0) \geqslant \sqrt{\Theta_0^2\cos^2(\theta^0) + \sin^2(\theta^0)}\,.$$

Since θ^0 goes to $\arcsin b_2$ when α goes to 0, we get

$$\liminf_{\alpha\to 0} \mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) = \liminf_{\alpha\to 0} \sigma(\theta^0) \geqslant \sqrt{(1-\Theta_0^2)b_2^2 + \Theta_0^2}\,.$$

Since $\Xi_0 \in (0,1)$ (see Subsection 5.2), we get:

$$\forall b_2 \in [0,1], \quad \Xi_0 b_2 < \sqrt{(1-\Theta_0^2)b_2^2 + \Theta_0^2}$$

and we deduce from Corollary 5.3:

$$\limsup_{\alpha\to 0} E(\mathbf{B}, \mathcal{W}_\alpha) < \liminf_{\alpha\to 0} \mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha)\ .$$

The theorem follows. □

Remark 5.5. It is possible to use gaussian quasimodes in (5.3) and to deduce for $E(\mathbf{B}, \mathcal{W}_\alpha)$ a polynomial in α upper bound with explicit constants (see [40, Section 6.3]). This allows to get analytic value of $\alpha(\mathbf{B})$, for example we get with numerical approximations $\alpha(\mathbf{B}) \geqslant 0.38\pi$ for the magnetic field $\mathbf{B} = (0,1,0)$ normal to the plane of symmetry.

Remark 5.6. The previous theorem remains true in the special case $b_2 = 0$ (see [40, Section 7]) but the proof is different since the limit operator when α goes to 0 is not anymore the operator $\mathfrak{q}_\tau$ introduced in Section 5.1.

6. Numerical simulations

Let $C := (0,L)^2$ be the square of length L. We perform a rotation by $-\frac{\pi}{4}$ around the origin and the scaling $X_2 := x_2 \tan\frac{\alpha}{2}$ along the x_2-axis. The image of C by these transformations is a rhombus of opening α denoted by $\mathrm{R}(\alpha, \mathrm{L})$. The length of the diagonal supported by the x_1-axis is $\sqrt{2}L$. Using the finite element library Mélina ([34]), we compute the first eigenpairs of $(-i\nabla - \mathbf{A}^{\|})^2 + V^{\tau}_{\mathbf{B}^\perp}$ on $\mathrm{R}(\alpha, \mathrm{L})$ with a Dirichlet condition on the artificial boundary $\{\partial\mathrm{R}(\alpha,\mathrm{L}) \cap \{\mathrm{x}_1 > \frac{1}{\sqrt{2}}\mathrm{L}\}$. We denote by $\breve{s}(\mathbf{B}, \mathcal{S}_\alpha; \tau)$ the numerical approximation of the first eigenvalue of this operator. For L large, $\breve{s}(\mathbf{B}, \mathcal{S}_\alpha; \tau)$ is a numerical approximation of $s(\mathbf{B}, \mathcal{S}_\alpha; \tau)$. We refer to [40, Annex C and Chapter 5] for more details about the meshes and the degree of the approximations we have used.

We make numerical simulations for the magnetic field $\mathbf{B} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ which is normal to the edge. An associated linear potential is $\mathbf{A}(x_1,x_2,x_3) = (0,0,-\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}})$ and we have

$$H(\mathbf{A}^{\|}, \mathcal{S}_\alpha) + V^{\tau}_{\mathbf{B}^\perp} = -\Delta + (\tfrac{x_2}{\sqrt{2}} - \tfrac{x_1}{\sqrt{2}} - \tau)^2\ .$$

We notice that in that case the reduced operator on $\mathcal{S}_\alpha$ is real and therefore its eigenfunctions have real values. For numerical simulations of eigenfunction with complex values, see [42, Section 7].

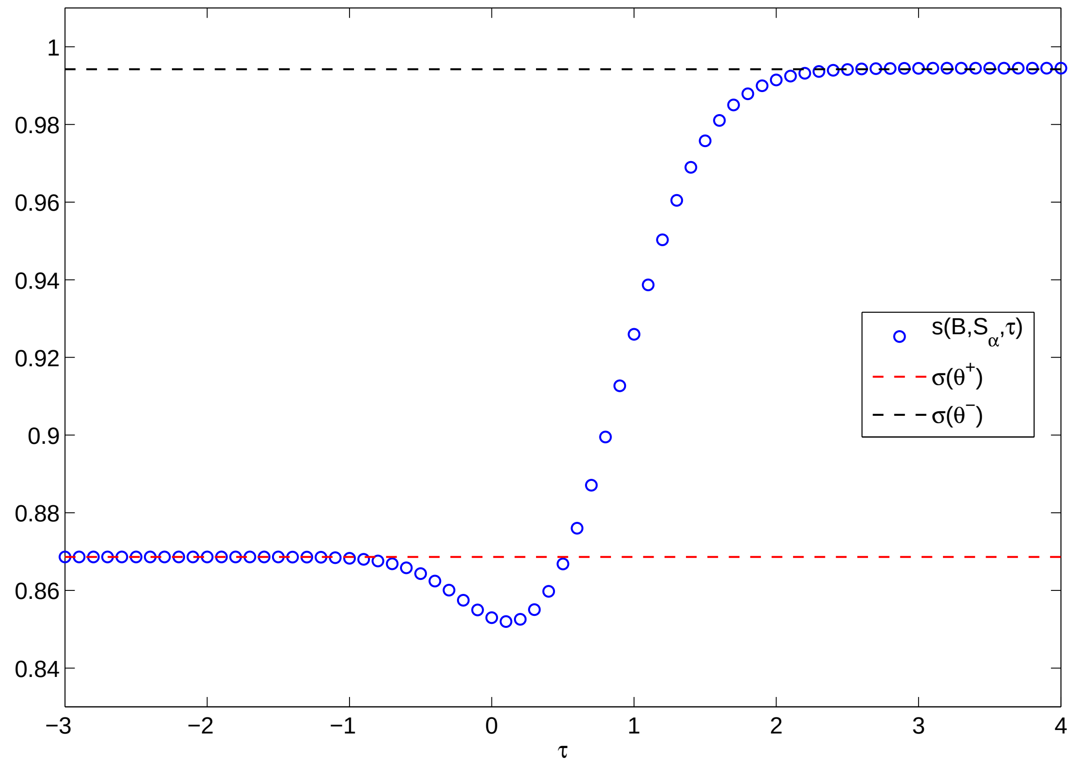

FIGURE 1. Magnetic field: $\mathbf{B} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. Opening angle: $\alpha = \frac{4\pi}{5}$. The numerical approximation of $s(\mathbf{B}, \mathcal{S}_\alpha; \tau)$ versus τ compared with $\sigma(\theta^+)$ and $\sigma(\theta^-)$.

On Figure 1 we have set $\alpha = \frac{4\pi}{5}$: the magnetic field is ingoing. In that case we have $\theta^+ = \frac{3\pi}{20}$ and $\theta^- = \frac{7\pi}{20}$. We have shown $\breve{s}(\mathbf{B}, \mathcal{S}_\alpha; \tau)$ for $\tau = \frac{k}{10}$ with $-30 \leqslant k \leqslant 40$. We have also shown $\sigma(\theta^+)$ and $\sigma(\theta^-)$ where the numerical approximations of $\sigma(\cdot)$ comes from [13]. $\breve{s}(\mathbf{B}, \mathcal{S}_\alpha; \tau)$ seems to converge to $\sigma(\theta^{\mp})$ when τ goes to $\pm\infty$ in agreement with Proposition 3.1. Moreover $\tau \mapsto \breve{s}(\mathbf{B}, \mathcal{S}_\alpha; \tau)$ reaches its infimum and this infimum is strictly below $\sigma(\theta^+) = \mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha)$. Therefore we think that inequality (1.7) is strict for these values of $\mathbf{B}$ and α.

On figure 2 we show normalized eigenfunctions of $(-i\nabla - \mathbf{A}^{\|}) + V^\tau_{\mathbf{B}^\perp}$ on $\mathrm{R}(\frac{4\pi}{5}, 20)$ associated with $\breve{s}(\mathbf{B}, \mathcal{S}_{\frac{4\pi}{5}}; \tau)$ for $\tau = k$, $-3 \leqslant k \leqslant 4$. We see that the eigenfunctions are localized near the line Υ where the potential $V^\tau_{\mathbf{B}^\perp}$ vanishes.

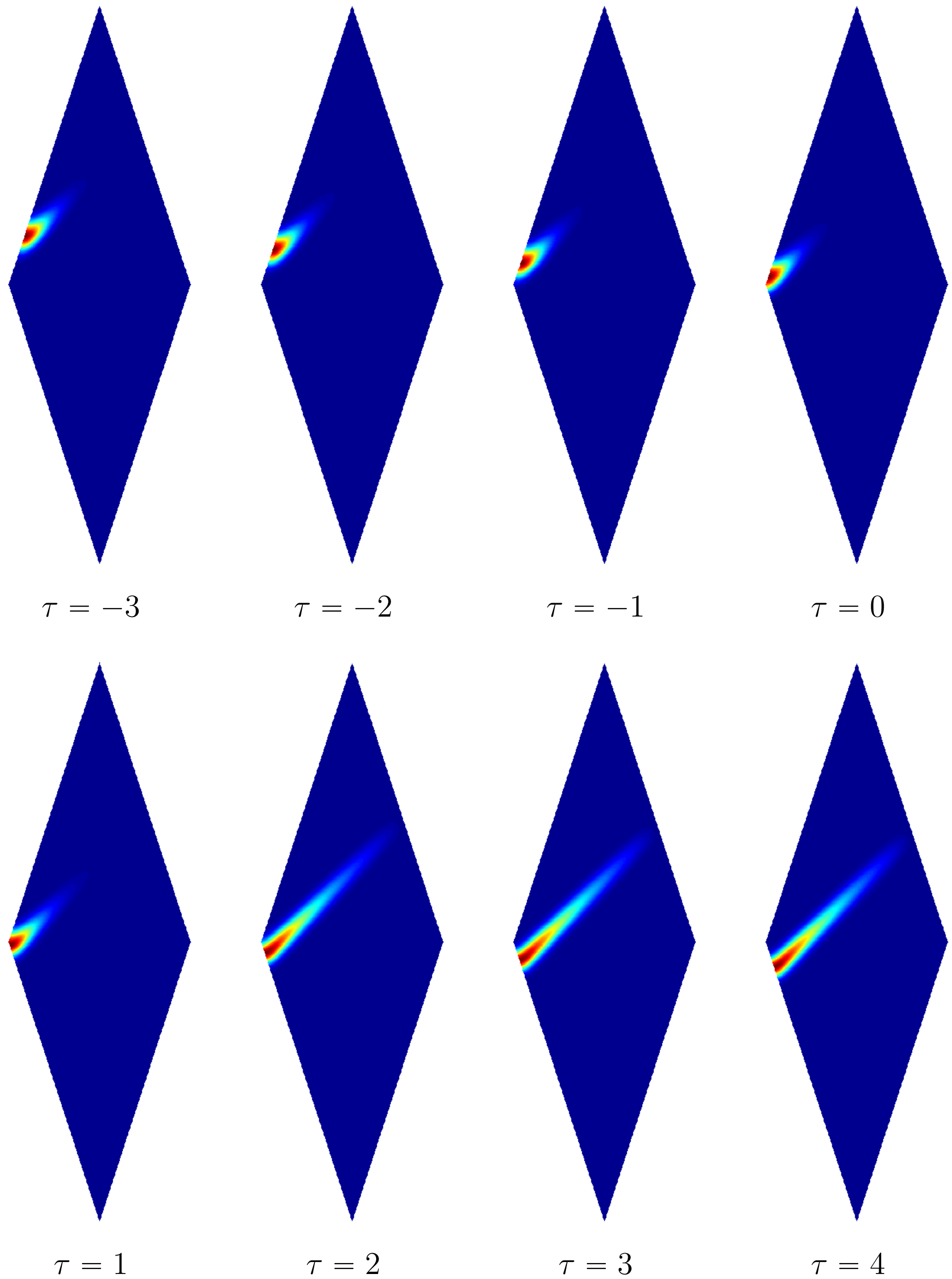

FIGURE 2. Magnetic field: $\mathbf{B} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. Opening angle: $\alpha = \frac{4\pi}{5}$. Normalized Eigenvectors of $H(\mathbf{A}^{\parallel}, \mathcal{S}_\alpha) + V^{\tau}_{\mathbf{B}^\perp}$ associated with $s(\mathbf{B}, \mathcal{S}_\alpha; \tau)$. From top to bottom and left to right: $\tau = k$, $-3 \leqslant k \leqslant 4$. Computational domain: $\mathrm{R}(20, \frac{4\pi}{5})$.

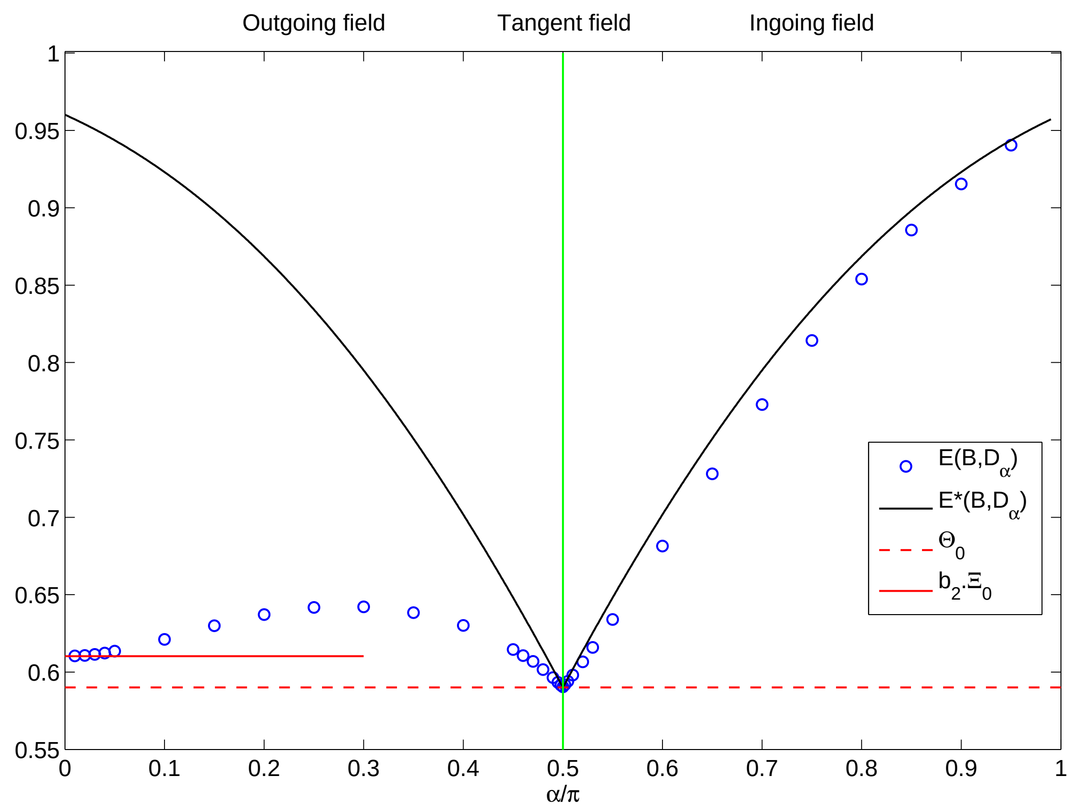

FIGURE 3. Magnetic field: $\mathbf{B} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. The numerical approximation of $E(\mathbf{B}, \mathcal{W}_\alpha)$ versus $\frac{\alpha}{\pi}$ compared with $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha)$, $b_2\Xi_0$ and Θ_0.

On figure 3 we show numerical approximations of $E(\mathbf{B}, \mathcal{W}_\alpha)$. For each value of α we compute $s(\mathbf{B}, \mathcal{S}_\alpha; \tau)$ for several values of τ and we define

$$\breve{E}(\mathbf{B}, \mathcal{W}_\alpha) := \inf_\tau \breve{s}(\mathbf{B}, \mathcal{S}_\alpha; \tau)$$

a numerical approximation of $E(\mathbf{B}, \mathcal{W}_\alpha)$. The magnetic field is outgoing when $\alpha \in (0, \frac{\pi}{2})$, ingoing when $\alpha \in (\frac{\pi}{2}, \pi)$ and tangent when $\alpha = \frac{\pi}{2}$. We notice that $\breve{E}(\mathbf{B}, \mathcal{W}_\alpha)$ seems to converge to $b_2\Xi_0$ (see Subsection 5.2). We have also plotted $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha)$ according to (1.6) and to the numerical values of $\sigma(\cdot)$ coming from [13]. We see that for $\alpha \neq \frac{\pi}{2}$, we have $\breve{E}(\mathbf{B}, \mathcal{W}_\alpha) < \mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha)$ whereas $\breve{E}(\mathbf{B}, \mathcal{W}_{\frac{\pi}{2}}) \approx \Theta_0 = \mathscr{E}^*(\mathbf{B}, \mathcal{W}_{\frac{\pi}{2}})$. Let us also notice that $\alpha \mapsto E(\mathbf{B}, \mathcal{W}_\alpha)$ seems not to be $\mathcal{C}^1$ in $\alpha = \frac{\pi}{2}$.

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LABORATOIRE CPT, UMR 7332 DU CNRS, CAMPUS DE LUMINY, 13288 MARSEILLE CEDEX 9, FRANCE,
E-mail address: NICOLAS.POPOFF@CPT.UNIV-MRS.FR